第五章 雙變量隨機變數

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連續型隨機變數的例子

Ex 1(例 1.4) X_1, X_2 are independent (random sample) from the *pdf*

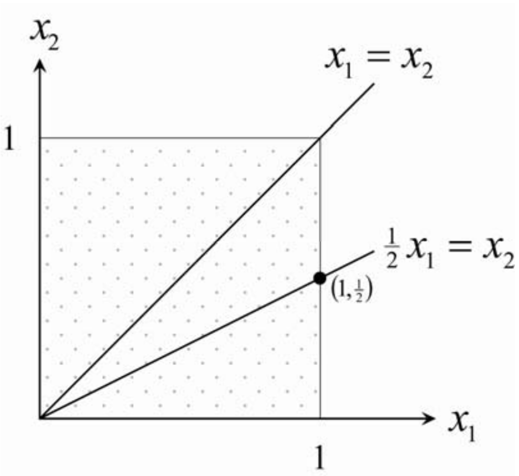
$$f(x)=\begin{cases} 2x & 0<x<1 \\ 0 & o.w. \end{cases}$$

Find $P(X_1 < X_2 | X_1 < 2X_2)$

$\Rightarrow$

The joint pdf of X_1, X_2 is $f(x_1, x_2)=\begin{cases} 4x_1x_2 & 0<x_1<1,\ 0<x_2<1 \\ 0 & o.w. \end{cases}$

$$P(X_1 < X_2 | X_1 < 2X_2) = \frac{P(X_1 < X_2, X_1 < 2X_2)}{P(X_1 < 2X_2)} = \frac{P(X_1 < X_2)}{P(X_1 < 2X_2)}$$



$$P\left(X_1 < 2X_2\right) = \int_0^1 \int_{\frac{x_1}{2}}^1 4x_1x_2\,dx_2\,dx_1 = 4\int_0^1 \left[\frac{1}{2}x_1x_2^2\right]\Bigg|_{x=\frac{x_1}{2}}^{x_2=1} dx_1$$

$$= 2\int_0^1 \left(x_1 - \tfrac{1}{4}x_1^3\right)dx_1 = 2\left[\tfrac{1}{2}x_1^2 - \tfrac{1}{16}x_1^4\right]_0^1 = 2\left(\tfrac{1}{2} - \tfrac{1}{16}\right) = 1 - \tfrac{1}{8} = \tfrac{7}{8}$$

$$P\left(X_1 < X_2\right) = \int_0^1 \int_{x_1}^1 4x_1x_2\,dx_2\,dx_1 = 4\int_0^1 \tfrac{1}{2}x_1x_2^2\Big|_{x_2=x_1}^{x_2=1} dx_1 = 2\int_0^1 (x_1 - x_1^3)\,dx_1$$

$$= 2\left(\tfrac{1}{2}x_1^2 - \tfrac{1}{4}x_1^4\right)\Big|_0^1 = 2\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2}$$

$$\therefore P\left(X_1 < X_2 \middle| X_1 < 2X_2\right) = \frac{1/2}{7/8} = \frac{4}{7}$$

Ex 2(例 1.6) (Y_1, Y_2) with joint *pdf* $f(y_1, y_2) = \begin{cases} c(y_1 + y_2^2) & 0 \le y_1 \le 1 \\ & 0 \le y_2 \le 1 \\ 0 & o.w. \end{cases}$

1. Find the marginal *pdf* of Y_2
2. Find $f(y_1 | y_2)$
3. Find $P(y_1 \le \frac{1}{2} | y_2 = \frac{1}{2})$

$\Rightarrow$

1. $\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 dy_1 = 1 \Rightarrow c\int_0^1\int_0^1 (y_1 + y_2^2)\, dy_2 dy_1 = 1$

$$\Rightarrow c\int_0^1\int_0^1 \left(y_1 y_2 + \tfrac{1}{3} y_2^3\right)\Big|_{y_2=0}^{y_2=1} dy_1 = 1 \Rightarrow c\int_0^1 (y_1 + \tfrac{1}{3}) dy_1 = 1 \Rightarrow c\left(\tfrac{1}{2} y_1^2 + \tfrac{1}{3} y_1\right)\Big|_0^1 = 1$$

$$\Rightarrow c\left(\tfrac{1}{2}+\tfrac{1}{3}\right)=1 \Rightarrow c=\tfrac{6}{5}$$

The marginal *pdf* of y_2 is $f_{Y_2}(y_2)=\int_{-\infty}^{\infty} f(y_1,y_2)dy_1 = \tfrac{6}{5}\int_0^1 \left(y_1+y_2^2\right)dy_1$

$$=\tfrac{6}{5}\left(\tfrac{1}{2}y_1^2+y_2^2y_1\right)\Big|_{y_1=0}^{y_1=1}=\tfrac{6}{5}\left(\tfrac{1}{2}+y_2^2\right)$$

$$\therefore \quad f_{Y_2}(y_2)=\begin{cases}\tfrac{6}{5}\left(\tfrac{1}{2}+y_2^2\right) & 0\le y_2\le 1\\ 0 & o.w.\end{cases}$$

2. $$f(y_1|y_2) = \frac{f(y_1, y_2)}{f(y_2)} = \frac{\frac{6}{5}\left(y_1 + y_2^2\right)}{\frac{6}{5}\left(\frac{1}{2} + y_2^2\right)} = \frac{y_1 + y_2^2}{\frac{1}{2} + y_2^2}$$

$$\therefore \quad f(y_1|y_2) = \begin{cases} \dfrac{y_1 + y_2^2}{\frac{1}{2} + y_2^2} & 0 \le y_1 \le 1 \quad 0 \le y_2 \le 1 \\ 0 & o.w. \end{cases}$$

3. $\because$ $$f\left(y_1 \middle| \tfrac{1}{2}\right) = \begin{cases} \dfrac{y_1 + \left(\frac{1}{2}\right)^2}{\frac{1}{2} + \left(\frac{1}{2}\right)^2} & 0 \le y_1 \le 1 \\ 0 & o.w. \end{cases}$$

$$P\left(y_1 \le \tfrac{1}{2} \middle| y_2 = \tfrac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{y_1 + \left(\frac{1}{2}\right)^2}{\frac{1}{2} + \left(\frac{1}{2}\right)^2} dy_1 = \int_0^{\frac{1}{2}} \tfrac{4}{3}\left(y_1 + \tfrac{1}{4}\right) dy_1$$

$$= \tfrac{4}{3}\left[\frac{1}{2} y_1^2 + \tfrac{1}{4} y_1\right]_0^{\frac{1}{2}} = \frac{1}{3}$$

Ex 3(例 1.20) (X,Y) with joint *pdf* $f(x,y)=\begin{cases}1 & 0\le x\le 1, 0\le y\le 1\\ 0 & o.w.\end{cases}$

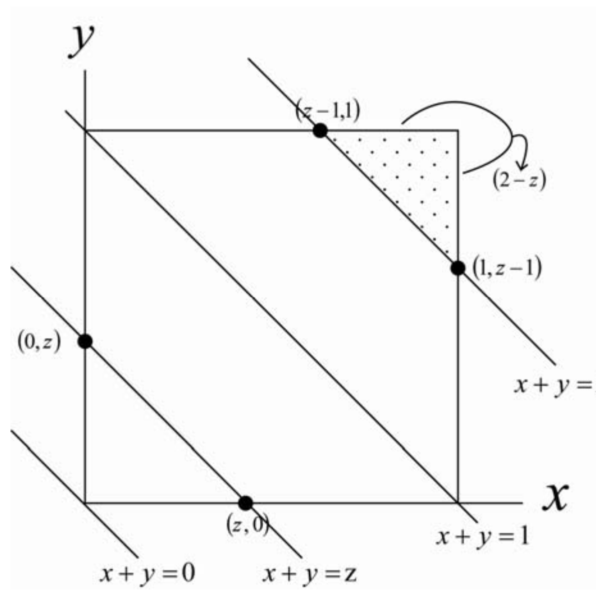
$Z=X+Y$, find the *pdf* of Z.

$\Rightarrow$

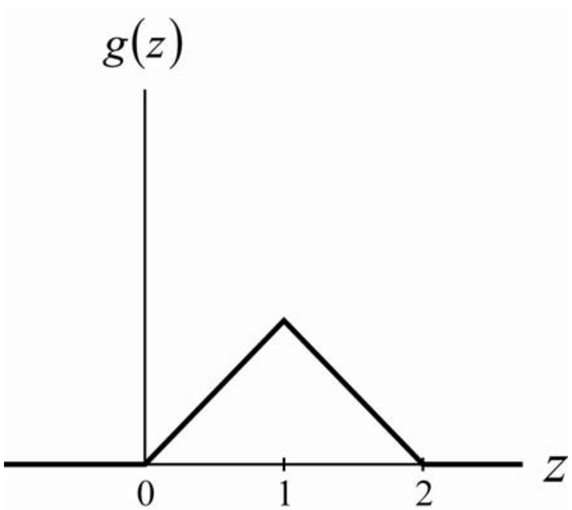
The *cdf* of Z is

$G(z)=P(Z\le z)=P(x+y\le z)$

$$=\begin{cases}0 & if\ z\le 0\\ \frac{1}{2}z^2 & if\ 0<z\le 1\\ 1-\frac{1}{2}(2-z)^2 & if\ 1<z\le 2\\ 1 & if\ z>2\end{cases}$$



$$\therefore \text{ The pdf of } Z \text{ is } g(z) = \frac{dG(z)}{dz} = \begin{cases} z & 0 \le z \le 1 \\ z - 2 & 1 < z \le 2 \\ 0 & o.w. \end{cases}$$



Ex 4(例 1.22) (X_1, X_2) are random sample from the *pdf*

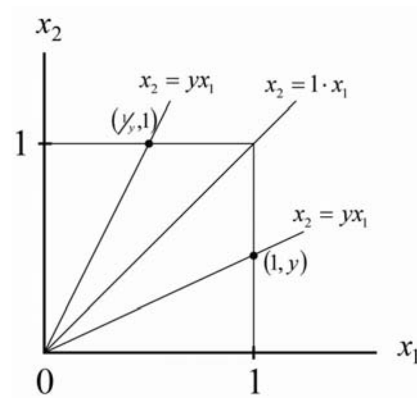
$$f(x)=\begin{cases}1 & 0\le x\le 1\\ 0 & o.w.\end{cases}$$ 令 $Y=\dfrac{X_2}{X_1}$,find the pdf of Y.

$\Rightarrow$

The joint pdf of (X_1, X_2) is $f(x_1,x_2)=\begin{cases}1 & 0\le x_1\le 1, 0\le x_2\le 1\\ 0 & o.w.\end{cases}$

The cdf of Y is

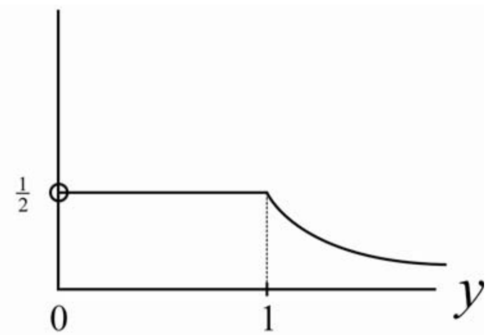
$$F_Y(y)=P(Y\le y)=P\left(\tfrac{X_2}{X_1}\le y\right)$$



$$
= \begin{cases} if\ \ y < 0 & , \qquad 0 \\ if\ \ 0 < y \le 1 & , \qquad \tfrac{1}{2} \cdot 1 \cdot y \\ if\ \ 1 < \mathrm{y} & , \quad 1 - \tfrac{1}{2}(1)\left(\tfrac{1}{y}\right) = 1 - \tfrac{1}{2y} \end{cases}
$$

$$
\therefore \quad F_Y(y) = \begin{cases} 0 & , \quad if\ \ y < 0 \\ \tfrac{1}{2} y & , \quad if\ \ 0 < y \le 1 \\ 1 - \tfrac{1}{2y} & , \quad if\ \ y > 1 \end{cases}
$$

The *pdf* of Y is $f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{2} & 0 < y \le 1 \\ \frac{1}{2y^2} & y > 1 \\ 0 & o.w. \end{cases}$



Ex 5(例 1.30)　假設 X,Y joint *pdf* 如下：

$$f(x,0)=\lambda_1 e^{-(\lambda_1+\lambda_2)x} \qquad x>0,\quad \lambda_1,\lambda_2>0$$

$$f(x,1)=\lambda_2 e^{-(\lambda_1+\lambda_2)x} \qquad x>0,\quad \lambda_1,\lambda_2>0$$

1. Find marginal pdf of X,Y.
2. Find $E(X),E(Y)$
3. Find $Var(X),Var(Y),Cov(X,Y)$

$\Rightarrow$

1. The marginal *pdf* of *X* is

$$f_X(x)=\sum_{y=0,1} f(x,y)=(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x} \qquad x>0$$

The marginal *pdf* of *Y* is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_{-\infty}^{\infty} \lambda_1 e^{-(\lambda_1+\lambda_2)x} dx & ,y=0 \\ \int_{-\infty}^{\infty} \lambda_2 e^{-(\lambda_1+\lambda_2)x} dx & ,y=1 \end{cases}$$

$$= \lambda_1 \cdot (1) \tfrac{1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)} \Big|_0^{\infty} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$\therefore \quad f_Y(y) = \begin{cases} \dfrac{\lambda_1}{\lambda_1+\lambda_2} & ,y=0 \\ \dfrac{\lambda_2}{\lambda_1+\lambda_2} & ,y=1 \end{cases}$$

2. $E(X)=\int_{-\infty}^{\infty} xf(x)dx=\int_{-\infty}^{\infty} x(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x}dx=(\lambda_1+\lambda_2)\int_{-\infty}^{\infty} xe^{-(\lambda_1+\lambda_2)x}dx$

令$u=(\lambda_1+\lambda_2)x \Rightarrow du=(\lambda_1+\lambda_2)dx$

$=(\lambda_1+\lambda_2)\int_0^{\infty}\frac{1}{\lambda_1+\lambda_2}u\cdot e^{-u}\cdot\frac{1}{\lambda_1+\lambda_2}du=\frac{1}{\lambda_1+\lambda_2}\int_0^{\infty}u\cdot e^{-u}du$

$=\frac{1}{\lambda_1+\lambda_2}\Gamma(2)=\frac{1}{\lambda_1+\lambda_2}$

(***Note*:** $\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx$)

$E(Y)=\sum_{y=0,1} yf_Y(y)=(1)\frac{\lambda_2}{\lambda_1+\lambda_2}=\frac{\lambda_2}{\lambda_1+\lambda_2}$

3. $E\left(X^2\right)=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_{-\infty}^{\infty} x^2(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x}dx$

令$u=(\lambda_1+\lambda_2)x \Rightarrow du=(\lambda_1+\lambda_2)dx$

$=\int_0^{\infty}\frac{u^2}{(\lambda_1+\lambda_2)^2}(\lambda_1+\lambda_2)e^{-u}\frac{du}{(\lambda_1+\lambda_2)}=(\lambda_1+\lambda_2)^{-2}\int_0^{\infty}u^2e^u du=(\lambda_1+\lambda_2)^{-2}\Gamma(3)$

$=(\lambda_1+\lambda_2)^{-2}2\Gamma(2)=\frac{2}{(\lambda_1+\lambda_2)^2}$

$\therefore\ Var(X)=E\left(X^2\right)-E(X)^2=\frac{2}{(\lambda_1+\lambda_2)^2}-\frac{1}{(\lambda_1+\lambda_2)^2}=\frac{1}{(\lambda_1+\lambda_2)^2}$

$$E\left(Y^2\right) = \sum_{y=0,1} y^2 f_Y(y) = (1)^2 \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$Var(Y) = E\left(Y^2\right) - E(Y)^2 = \frac{\lambda_2}{\lambda_1+\lambda_2} - \left(\frac{\lambda_2}{(\lambda_1+\lambda_2)}\right)^2 = \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$$

$$E(XY) = \int_0^\infty \sum_{y=0,1} xyf(x,y)dx = \int_0^\infty xf(x,1)dx = \int_0^\infty x\lambda_2 e^{-(\lambda_1+\lambda_2)x} dx$$

$$= \frac{\lambda_2}{(\lambda_1+\lambda_2)^2}$$

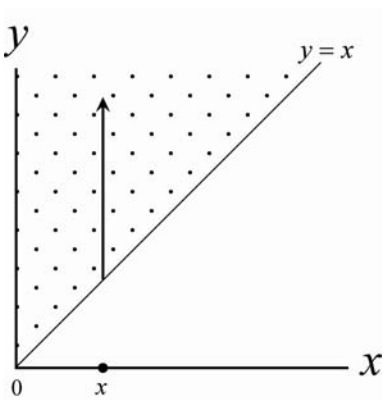
$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{\lambda_2}{(\lambda_1+\lambda_2)^2} - \left(\frac{1}{\lambda_1+\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right) = 0$$

Ex 6(例 3.18) $r.v.(X,Y)$ with joint *pdf* $f(x,y)=\begin{cases} ce^{-3y} & 0\le x\le y \\ 0 & o.w. \end{cases}$

1. Find c. 2. Find the conditional *pdf* of Y given $X=1$.

3. $E(Y|X=1)=?$ $Var(Y|X=1)=?$

$\Rightarrow$



1. $\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = 1$

$$\Rightarrow \int_{0}^{\infty}\int_{y=x}^{\infty} ce^{-3y}dydx = 1$$

$$\Rightarrow c\int_{0}^{\infty}\left(-\tfrac{1}{3}e^{-3y}\right)_{y=x}^{y=\infty}dx = 1 \Rightarrow \tfrac{c}{3}\int_{0}^{\infty} e^{-3x}dx = 1 \Rightarrow \left(\tfrac{c}{3}\right)\left(-\tfrac{1}{3}e^{-3x}\right)_{0}^{\infty} = 1$$

$$\Rightarrow c = 9$$

2. $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_x^{\infty} 9e^{-3y}dy = 9\cdot\left(-\tfrac{1}{3}e^{-3y}\right)\Big|_{y=x}^{y=\infty} = 3e^{-3x}, x \geq 0$$

$$i.e.\quad f_X(x) = \begin{cases} 3e^{-3x} & , \quad x \geq 0 \\ 0 & , \quad o.w. \end{cases}$$

$$\therefore\quad f_{Y|X}(y|x) = \frac{9e^{-3y}}{3e^{-3x}} = \begin{cases} 3e^{-3(y-x)} & , \quad 0 \leq x \leq y \\ 0 & , \quad o.w. \end{cases}$$

$$f_{Y|X}(y|1) = \frac{9e^{-3y}}{3e^{-3x}} = \begin{cases} 3e^{-3(y-1)} & , \quad y \geq 1 \\ 0 & , \quad o.w. \end{cases}$$

3. $E\left(Y\middle|X=1\right)=\int_{-\infty}^{\infty} y f_{Y|X}\left(y\middle|1\right)dy=\int_{1}^{\infty} y\cdot 3e^{-3(y-1)}dy$

令 $u=y-1\Rightarrow du=dy$

$$=3\int_{1}^{\infty}(u+1)e^{-3u}du=3\left[\int_{0}^{\infty} ue^{-3u}du+\int_{0}^{\infty} e^{-3u}du\right]$$

$$=3\left(-\tfrac{1}{3}\right)e^{-3u}\Big|_{0}^{\infty}+3\int_{0}^{\infty} ue^{-3u}du=1+3\int_{0}^{\infty} ue^{-3u}du$$

令 $U=3u\Rightarrow dU=3du$

$$=1+3\int_{0}^{\infty}\tfrac{1}{3}Ue^{-U}\left(\tfrac{1}{3}\right)dU=1+\tfrac{1}{3}\int_{0}^{\infty} Ue^{-U}dU=1+\tfrac{1}{3}\Gamma(2)=1+\tfrac{1}{3}=\tfrac{4}{3}$$

$$E\left(Y^2 \middle| X=1\right)=\int_{-\infty}^{\infty} y^2 f_{Y|X}\left(y|1\right)dy=\int_1^{\infty} y^2\cdot 3e^{-3(y-1)}dy$$

令 $u=3(y-1)\Rightarrow dy=\frac{1}{3}du$

$$=\int_0^{\infty}\left(\tfrac{u+3}{3}\right)^2\cdot 3e^{-u}\left(\tfrac{1}{3}\right)du=\tfrac{1}{9}\int_0^{\infty}(u+3)^2 e^{-u}du$$

$$=\tfrac{1}{9}\int_0^{\infty}\left(u^2+6u+9\right)e^{-u}du=\tfrac{1}{9}\left(\Gamma(3)+6\Gamma(2)+9\Gamma(0)\right)$$

$$=\tfrac{1}{9}(2+6+9)=\tfrac{17}{9}$$

$$Var\left(Y \middle| X=1\right)=E\left(Y^2 \middle| X=1\right)-E\left(Y \middle| X=1\right)^2=\tfrac{17}{9}-\left(\tfrac{4}{3}\right)^2=\tfrac{1}{9}$$