第五章 雙變量隨機變數

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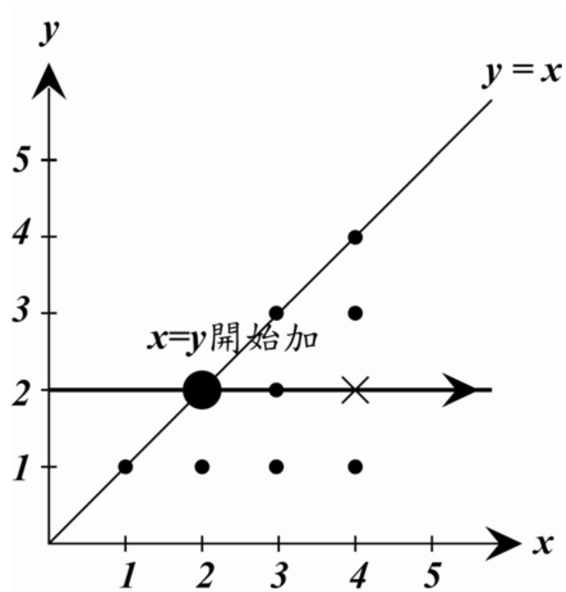
Ex 11(例 3.1) X,Y 其 joint *pdf* 為 $f(x,y)=\begin{cases}\frac{2}{n(n+1)} & y=1,2,\cdots x\\ 0 & x=1,2,\cdots n\end{cases}$

1. Find $E(Y|X)$ $E(X|Y)$

2. Find $\rho_{X,Y}$



$\Rightarrow$

1. $E(Y|X)=\sum_y y f_{Y|X}(y|x)$

$$f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x)=\sum_y f_{X,Y}(x,y)=\sum_{y=1}^{x}\frac{2}{n(n+1)}=\frac{2x}{n(n+1)}\qquad x=1,2,\cdots n$$

$$\therefore\quad f_{Y|X}(y|x)=\begin{cases}\dfrac{\frac{2}{n(n+1)}}{\frac{2x}{n(n+1)}}=\dfrac{1}{x} & \begin{matrix}y=1,2,\cdots x\\ x=1,2,\cdots n\end{matrix}\\ 0 & o.w.\end{cases}$$

$$\therefore\quad E(Y|X)=\sum_y y f_{Y|X}(y|x)=\sum_{y=1}^{x} y\cdot\tfrac{1}{x}=\tfrac{1}{x}\cdot\tfrac{x(x+1)}{2}=\tfrac{x+1}{2}\qquad x=1,2,\cdots n$$

同理, $f_Y(y)=\sum_x y f_{X,Y}(x,y)=\sum_{x=y}^{n}\frac{2}{n(n+1)}\qquad y=1,2,\cdots x$

$$=\begin{cases}\frac{2}{n(n+1)}\cdot(n-y+1) & y=1,2,\cdots x\\ 0 & o.w.\end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{\frac{2}{n(n+1)}}{\frac{2(n-y+1)}{n(n+1)}} = \frac{1}{(n-y+1)} & x = y, y+1, \cdots n \\ 0 & o.w. \end{cases}$$

$$E(X|Y) = \sum_x x f_{X|Y}(x|y) = \sum_{x=y}^{n} x \cdot \frac{1}{n-y+1} = \frac{1}{n-y+1} \cdot \frac{(y+n)(n-Y+1)}{2} = \frac{y+n}{2} \qquad x = 1, 2, \cdots n$$

2. $\because$ $E(Y|X) = \left(\frac{x+1}{2}\right) = \frac{1}{2} + \left(\frac{1}{2}\right)x$

$E(X|Y) = \left(\frac{y+n}{2}\right) = \left(\frac{n}{2}\right) + \left(\frac{1}{2}\right)y$ 為 Linear

Note：$E(Y|X) = a + bX = a + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)x$

$E(X|Y) = a' + b'X = a' + \rho\left(\frac{\sigma_X}{\sigma_Y}\right)y$

$$\rho > 0 \quad \& \quad \rho^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \rho = \frac{1}{2}$$

另解：

$$\because \quad \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}}$$

$$E(XY) = \sum_{x=1}^{n}\sum_{y=1}^{x} xyf_{X,Y}(x,y) = \sum_{x=1}^{n}\sum_{y=1}^{x} xy\cdot\frac{2}{n(n+1)} = \frac{2}{n(n+1)}\sum_{x=1}^{n} x\sum_{y=1}^{x} y$$

$$= \frac{2}{n(n+1)}\sum_{x=1}^{n} x\cdot\frac{x(x+1)}{2} = \frac{2}{n(n+1)}\sum_{x=1}^{n} x^2(x+1) = \frac{1}{n(n+1)}\left[\sum_{x=1}^{n} x^3 + \sum_{x=1}^{n} x^2\right]$$

$$= \frac{1}{n(n+1)}\left[\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right] = \frac{n(n+1)}{4} + \frac{(2n+1)}{6}$$

$$= \frac{3n^2 + 7n + 2}{12} = \frac{(n+2)(3n+1)}{12}$$

$$E(X) = \sum_{x=1}^{n} x f_X(x) = \sum_{x=1}^{n} x \cdot \frac{2x}{n(n+1)} = \frac{2}{n(n+1)} \sum_{x=1}^{n} x^2 = \frac{2}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2n+1}{3}$$

$$\because \quad E(Y|X) = \frac{1}{2} + \frac{1}{2}X$$

$$\therefore EE(Y|X) = E\left(\frac{1}{2} + \frac{1}{2}X\right)$$

$\Rightarrow$

$$E(Y) = \tfrac{1}{2} + \tfrac{1}{2}E(X) = \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{2n+1}{3} = \tfrac{n+2}{3}$$

$$\therefore \; Cov(X,Y) = E(XY) - E(X)E(Y) = \tfrac{(n+2)(3n+1)}{12} - \left(\tfrac{2n+1}{3}\right)\left(\tfrac{n+2}{3}\right) = \left(\tfrac{n+2}{3}\right)\cdot\left(\tfrac{n-1}{12}\right)$$

$$E(X^2) = \sum_{x=1}^{n} x^2 f_X(x) = \sum_{x=1}^{n} x^2 \cdot \tfrac{2x}{n(n+1)} = \tfrac{2}{n(n+1)}\sum_{x=1}^{n} x^3 = \tfrac{2}{n(n+1)}\cdot\tfrac{n^2(n+1)^2}{4} = \tfrac{n(n+1)}{2}$$

$$\therefore \; Var(X) = E(X^2) - E(X)^2 = \tfrac{n(n+1)}{2} - \left(\tfrac{2n+1}{3}\right)^2 = \tfrac{n^2+n}{2} - \tfrac{4n^2+4n+1}{9} = \tfrac{(n+2)(n-1)}{18}$$

$$E\left(Y^2\right)=\sum_{y=1}^{n} y^2 f_Y(y)=\sum_{y=1}^{n} y^2\cdot\tfrac{2}{n(n+1)}(n-y+1)=\tfrac{2}{n(n+1)}\sum_{y=1}^{n} y^2(n-y+1)$$

$$=\tfrac{2}{n(n+1)}\left[(n+1)\sum_{y=1}^{n} y^2-\sum_{y=1}^{n} y^3\right]=\tfrac{2}{n(n+1)}\left[(n+1)\cdot\tfrac{n(n+1)(2n+1)}{6}-\tfrac{n^2(n+1)^2}{4}\right]$$

$$=\tfrac{n^2+3n+2}{6}=\tfrac{(n+1)(n+2)}{6}$$

$$\therefore\ Var(Y)=E\left(Y^2\right)-E(Y)^2=\tfrac{(n+1)(n+2)}{6}-\left(\tfrac{n+2}{3}\right)^2=\tfrac{(n+2)}{3}\cdot\left[\tfrac{n+1}{3}-\tfrac{n+2}{3}\right]$$

$$=\left(\tfrac{n+2}{3}\right)\left(\tfrac{3n+3-2n-4}{6}\right)=\left(\tfrac{n+2}{3}\right)\left(\tfrac{n-1}{6}\right)$$

$$\therefore\ \rho_{X,Y}=\frac{\left(\frac{n+2}{3}\right)\left(\frac{n-1}{12}\right)}{\sqrt{\frac{(n+2)(n-1)}{18}\cdot\frac{(n-1)(n+2)}{18}}}=\frac{\left(\frac{n+2}{3}\right)\left(\frac{n-1}{12}\right)}{\frac{(n+2)(n-1)}{18}}=\frac{\frac{1}{36}}{\frac{1}{18}}=\frac{1}{2}$$

Ex 12(例 3.3) (X,Y) 的聯合機率函數為

$$f(X,Y)=\begin{cases} k\left(x+y^2\right) & , \quad x=1,3, \quad y=-1,1,2 \\ 0 & , \quad o.w. \end{cases}$$

1. 求 k
2. $E(Y|X=1), Var(Y|X=1)$
3. $W=Min\{X,Y\}$機率分配

$\Rightarrow$

1. $\because \sum_{x=1,3}\sum_{y=-1,1,2} k\left(x+y^2\right)=1$

$$k\left[(1+1)+(1+1)+\left(1+2^2\right)+(3+1)+\left(3+1^2\right)+\left(3+2^2\right)\right]=1$$

$$k[24]=1 \quad \Rightarrow \quad k=\tfrac{1}{24}$$

∴ 聯合機率函數為：

$x \backslash y$	-1	1	2	$f_X(x)$
1	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{5}{24}$	$\frac{9}{24}$
3	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{7}{24}$	$\frac{15}{24}$
$f_Y(y)$	$\frac{6}{24}$	$\frac{6}{24}$	$\frac{12}{24}$	1

2. $E(Y|X=1) = \sum_y y f_{Y|X}(y|x=1)$

$$f_{Y|X}(y|x=1) = \frac{f_{X,Y}(x=1,y)}{f_X(x=1)} = \frac{f_{X,Y}(x=1,y)}{\frac{9}{24}} = \begin{cases} \frac{2/24}{9/24} & , \quad y=-1 \\ \frac{2/24}{9/24} & , \quad y=1 \\ \frac{5/24}{9/24} & , \quad y=2 \end{cases}$$

$$f_{Y|X}(y|x=1)=\begin{cases}\frac{2}{9} & , \quad y=-1\\ \frac{2}{9} & , \quad y=1\\ \frac{5}{9} & , \quad y=2\end{cases}$$

$$\therefore \quad E(Y|1)=\sum_y y f_{Y|X}(y|1)=(-1)\left(\tfrac{2}{9}\right)+(1)\left(\tfrac{2}{9}\right)+(2)\left(\tfrac{5}{9}\right)=\tfrac{10}{9}$$

$$Var(Y|1)=E(Y^2|1)-(E(Y|1))^2={}^{24}\!/_{9}$$

3.

w	-1	1	2
$f(w)$	$6/24$	$11/24$	$7/24$

Ex 13(例 3.8)　令 $T=\sum_{i=1}^{N} X_i$,　N is a *r.v.* & $E(N)<\infty$

$E(X_i)=E(X)\quad \forall i\quad \& X_i \coprod N$

1. $E(T)=E(N)E(X)$

2. $Var(T)=E(X)^2 Var(N)+E(N)Var(X)$

$\Rightarrow$

1. $\because\ E(T)=EE(T|N)=EE\left(\sum_{i=1}^{N} X_i \middle| N\right)=E\left(\sum_{i=1}^{N} E(X_i)\right)$

$$=E(NE(X))=E(N)E(X)$$

2. $\because\ Var(T)=EVar(T|N)+Var(E(T|N))$

$$Var(T|N) = Var\left(\sum_{i=1}^{N} X_i \middle| N\right) = \sum_{i=1}^{N} X_i Var(X_i|N) = \sum_{i=1}^{N} Var(X_i) = NVar(X)$$

$$\therefore \quad E(Var(T|N)) = E(NVar(X)) = E(N)Var(X)$$

$$Var(E(T|N)) = Var\left(E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right) = Var\left(\sum_{i=1}^{N} E(X_i|N)\right) = Var(NE(X))$$

$$= E(X)^2 Var(N)$$

$$\therefore \quad Var(T) = E(X)^2 Var(N) + E(N)Var(X)$$

Ex 14(例 3.15) X,Y *r.v.*s $E(X) = \mu_X \quad Var(X) = 4$
$E(Y) = 0 \quad Var(Y) = \sigma_y^2$

若 $E(X|Y) = 2Y^2, E(Y|X) = -3 + 0.5X$，計算 $\mu_X, \sigma_y^2, Cov(X, Y)$

$\Rightarrow$

$$E(Y) = EE(Y|X) = E(-3 + 0.5X) = -3 + 0.5\mu_X$$

$$\Rightarrow \quad \mu_X = 3/0.5 = 6 \qquad \therefore E(X) = 6$$

$$\because \quad E(X|Y) = 2Y^2$$

$$EE(X|Y) = E(2Y^2)$$

$$\Rightarrow E(X) = 2E(Y^2) \Rightarrow 6 = 2E(Y^2) \Rightarrow E(Y^2) = 3$$

$$Var(Y) = E(Y^2) - E(Y)^2 = 3 - 0 = 3 = \sigma_Y^2$$

$$\because \quad Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = EE(XY|X) = EXE(Y|X) = E(X(-3 + 0.5X))$$

$$= -3E(X) + 0.5E(X^2) = -3(6) + 0.5E(X^2)$$

$$= -18 + 0.5 \times 40 = 2$$

$$(\because \quad E(X^2) = Var(X) + E(X)^2 = 4 + 36 = 40)$$

$$\therefore \quad Cov(X,Y) = 2 - 3 \cdot 0 = 2$$

Ex 15(例 3.27) X,Y discrete r.v.s, 若 $E(Y|X)=X$ 且 $E(X|Y)=0$

1. $P(X=0)$ 2. $P(X=0)$

$\Rightarrow$

1. $E(X)=EE(X|Y)=E(0)=0$

$E(XY)=EE(XY|X)=EXE(Y|X)=E(X^2)$

而 $E(XY)=EE(XY|Y)=EYE(X|Y)=E(0)=0$

$\therefore\ E(X^2)=0\ \&\ E(X)=0\Rightarrow Var(X)=0$

X 為 degenerated. (退化分配)

$$\therefore \quad P(X = 0) = 1, P(X = 1) = 0$$

$$(\because E(X) = 0)$$

Ex 16(例 3.31)　X *r.v.* 機率函數為 $f_X(x)=P(X=x)=\begin{cases} p & ,x=-1 \\ q & ,x=0 \\ r & ,x=1 \end{cases}$

1. $E\left(X^2 \middle| X\right)$　2. $E\left(X \middle| X^2\right)$　3. $EE\left(X \middle| X^2\right)$

$\Rightarrow$

1. 令 $Y=X^2$

$$E(Y|X)=\sum_y y f_{Y|X}(y|x) \quad \text{而} \quad f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\because \begin{array}{lcl} x=-1 & \Rightarrow & y=x^2=1 \\ x=0 & \Rightarrow & y=0 \\ x=1 & \Rightarrow & y=1 \end{array}$$

$\therefore$

$x \backslash y$	0	1	$f_X(x)$
-1	0	p	p
0	q	0	q
1	0	r	r
$f_Y(y)$	q	$p+r$	

$$\therefore \quad f_{Y|X}\left(y \middle| x=-1\right)=\frac{f_{X,Y}\left(-1,y\right)}{f_X\left(-1\right)}=\begin{cases} 0/p=0 & if \quad y=0 \\ p/p=1 & if \quad y=1 \end{cases}$$

$$f_{Y|X}\left(y \middle| x=0\right)=\frac{f_{X,Y}\left(0,y\right)}{f_X\left(0\right)}=\begin{cases} q/q=1 & if \quad y=0 \\ 0/q=0 & if \quad y=1 \end{cases}$$

$$f_{Y|X}\left(y \middle| x=1\right)=\frac{f_{X,Y}\left(1,y\right)}{f_X\left(1\right)}=\begin{cases} 0/r=0 & if \quad y=0 \\ r/r=1 & if \quad y=1 \end{cases}$$

$$\therefore \quad E\left(Y \middle| X\right) = \sum_y y f_{Y|X}\left(y \middle| x\right) = \begin{cases} 0\text{x}0+1\text{x}1=1 & if \quad x=-1 \\ 0\text{x}1+1\text{x}0=0 & f \quad x=0 \\ 0\text{x}0+1\text{x}1=1 & f \quad x=1 \end{cases}$$

2. $E\left(X \middle| X^2\right) = E\left(X \middle| Y\right) = \sum_x x f_{X|Y}\left(x \middle| y\right)$

$$f_{X|Y}\left(x \middle| y=0\right) = \frac{f_{X,Y}\left(x,0\right)}{f_Y\left(0\right)} = \begin{cases} 0/q = 0 & if \quad x=-1 \\ q/q = 1 & f \quad x=0 \\ 0/q = 0 & f \quad x=1 \end{cases}$$

.

$$f_{X|Y}\left(x \middle| y=1\right)=\frac{f_{X,Y}(x,1)}{f_Y(1)}=\begin{cases} p/_{p+r} & if \quad x=-1 \\ 0 & f \quad x=0 \\ r/_{p+r} & f \quad x=1 \end{cases}$$

$$\therefore \quad E\left(X \middle| X^2\right)=E\left(X \middle| Y\right)=\sum_x x f_{X|Y}\left(x \middle| y\right)$$

$$=\begin{cases} (-1)\times 0+(0)\times 1+(1)\times 0=0 & if \ \ y=0 \\ (-1)\times p/_{p+r}+(0)\times 0+(1)\times\left(r/_{p+r}\right)= r-p/_{p+r} & if \ \ y=1 \end{cases}$$

3. $EE\left(X \middle| X^2\right)=EE\left(X \middle| Y\right)=E(X)=\sum_x x f(x)=-1\cdot p+0\cdot q+1\cdot r=r-p$