第五章 雙變量隨機變數

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Ex 1(例 1.1): Two *r.v.*s X, Y have the joint probability

$$P(i,j) = P(X = t_i, Y = u_j), \text{ the values are}$$

$$P(1,1) = 0.2 \quad P(1,2) = 0.2$$

$$P(2,1) = 0.1 \quad P(2,2) = 0.2$$

$$P(3,1) = 0.1 \quad P(3,2) = 0.3$$

1. Find $P(X = t_i), i = 1,2,3$ and $P(Y = u_j), j = 1,2$.
2. Are X and Y independent ?

$\Rightarrow$

1. The joint probability is

$x \backslash y$	1	2	$f_X(x)$	
1	0.2	0.1	0.3	→ $P(X=1)$
2	0.1	0.2	0.3	→ $P(X=2)$
3	0.1	0.3	0.4	→ $P(X=3)$
$f_Y(y)$	0.4	0.6	1	
	↓ $P(Y=1)$	↓ $P(Y=2)$		

2. $\because f_{X,Y}(1,1) = 0.2 \neq f_X(1) \cdot f_Y(1) = 0.3 \times 0.4 = 0.12$

$\therefore$ X and Y are NOT independent.

Ex2 (例 1.2) 一個袋子有 3 個白球, 2 個紅球與 3 個綠球, 隨機抽取兩球,令 X 表示白球個數, Y 表示紅球個數。

1. 求 X 與 Y 的聯合機率分配

2. Find $P(1 \le X + Y \le 2)$

3. Find $P(X = 1 | Y = 1)$

$\Rightarrow$

1. $P(X = x, Y = y) = \dfrac{C_x^3 C_y^2 C_{2-x-y}^3}{C_2^8}$ $\quad \begin{array}{c} x = 0,1,2 \\ y = 0,1,2 \\ 0 \le x + y \le 2 \end{array}$

2. $P\left(1 \le X + Y \le 2\right) = P\left(X + Y = 1\right) + P\left(X + Y = 2\right)$

$$= P\left(X = 1, Y = 0\right) + P\left(X = 0, Y = 1\right) + P\left(X = 0, Y = 2\right)$$
$$+ P\left(X = 1, Y = 1\right) + P\left(X = 2, Y = 0\right)$$

$$= \frac{C_1^3 C_0^2 C_1^3}{C_2^8} + \frac{C_0^3 C_1^2 C_1^3}{C_2^8} + \frac{C_0^3 C_2^2 C_0^3}{C_2^8} + \frac{C_1^3 C_1^2 C_0^3}{C_2^8} + \frac{C_2^3 C_0^2 C_0^3}{C_2^8}$$
$$= \frac{25}{28}$$

3. $$P(X=1|Y=1)=\frac{f_{X,Y}(1,1)}{f_Y(1)}=\frac{P(X=1,Y=1)}{P(Y=1)}=\frac{\frac{6}{28}}{\frac{12}{28}}=\frac{1}{2}$$

$$\because\quad P(Y=1)=\sum_x f_{X,Y}(x,1)=\sum_x \frac{C_x^3 C_y^2 C_{1-x}^3}{C_2^8}=\frac{C_0^3 C_1^2 C_1^3}{C_2^8}+\frac{C_1^3 C_1^2 C_0^3}{C_2^8}$$

$$=\frac{1}{C_2^8}(2\times 3+3\times 2)=\frac{12}{28}$$

Ex 3(例 1.14) $\left(X_1, X_2, X_3\right)$為 discrete joint *r.v.*s 其機率分配為

$f(0,0,0)=\frac{1}{8},\ f(0,0,1)=\frac{3}{8},\ f(0,1,1)=\frac{1}{8},\ f(1,0,0)=\frac{1}{8},$

$f(1,1,1)=\frac{1}{8},\ f(1,0,1)=\frac{1}{8}$

令$Y_1 = X_1 + X_2 + X_3, Y_2 = \left|X_3 - X_1\right|$ 求$\left(Y_1, Y_2\right)$聯合機率分配

$\Rightarrow$

$$Y_1 = \begin{cases} 0 & if\ (0,0,0), \frac{1}{8} \\ 1 & if\ (0,0,1), \frac{3}{8} \\ 2 & if\ (0,1,1), \frac{1}{8} \\ 1 & if\ (1,0,0), \frac{1}{8} \\ 3 & if\ (1,1,1), \frac{1}{8} \\ 2 & if\ (1,0,1), \frac{1}{8} \end{cases} \qquad Y_2 = \begin{cases} 0 & if\ (0,0,0), \frac{1}{8} \\ 1 & if\ (0,0,1), \frac{3}{8} \\ 1 & if\ (0,1,1), \frac{1}{8} \\ 1 & if\ (1,0,0), \frac{1}{8} \\ 0 & if\ (1,1,1), \frac{1}{8} \\ 0 & if\ (1,0,1), \frac{1}{8} \end{cases}$$

$\therefore$ Y_1的值 0,1,2,3

Y_2的值 0,1

Y_2 \ Y_1	0	1	2	3
0	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{8}$
1	0	$\frac{4}{8}$	$\frac{1}{8}$	0

Ex 4(例 1.16)　Suppose X, Y are independent with the probability functions. $P(X=x)=\dfrac{x}{6}$ $x=1,2,3.$ $P(Y=y)=\dfrac{y+2}{10}$, $y=-1,2,3.$

Using the mgf to find the distribution of $X-Y$.

$\Rightarrow$

The *mgf* of $X-Y$ is

$$M(t)=E\left(e^{t(X-Y)}\right)=E\left(e^{tX}\cdot e^{-tY}\right)=E\left(e^{tX}\right)E\left(e^{-tY}\right)=M_X(t)M_Y(-t)$$

$$M_X(t)=E\left(e^{tX}\right)=\sum_x e^{tx}f_X(x)=e^t f_X(1)+e^{2t}f_X(2)+e^{3t}f_X(3)$$

$$= \tfrac{1}{6}e^{t} + \tfrac{2}{6}e^{2t} + \tfrac{3}{6}e^{3t}$$

$$M_Y(t) = E\left(e^{tY}\right) = \sum_y e^{ty} f_Y(y) = e^{-t} f_X(-1) + e^{2t} f_X(2) + e^{3t} f_X(3)$$

$$= \tfrac{1}{10}e^{-t} + \tfrac{4}{10}e^{2t} + \tfrac{5}{10}e^{3t}$$

$$\therefore \quad M_Y(-t) = \tfrac{1}{10}e^{t} + \tfrac{4}{10}e^{-2t} + \tfrac{5}{10}e^{-3t}$$

$\Rightarrow$

$$M(t) = M_X(t)M_Y(-t) = \left(\tfrac{1}{6}e^{t} + \tfrac{2}{6}e^{2t} + \tfrac{3}{6}e^{3t}\right)\cdot\left(\tfrac{1}{10}e^{t} + \tfrac{4}{10}e^{-2t} + \tfrac{5}{10}e^{-3t}\right)$$

$$= \tfrac{5}{60}e^{-2t} + \tfrac{14}{60}e^{-t} + \tfrac{23}{60}e^{0t} + \tfrac{12}{60}e^{t} + \tfrac{1}{60}e^{2t} + \tfrac{2}{60}e^{3t} + \tfrac{3}{60}e^{4t}$$

Let $Z = X - Y$

$$= E\left(e^{Zt}\right) = \sum_z e^{zt} f_Z(z)$$

$\because$ mgf 唯一決定一種分配

$\therefore$

Z	-2	-1	0	1	2	3	4
$f_Z(z)$	$\frac{5}{60}$	$\frac{14}{60}$	$\frac{23}{60}$	$\frac{12}{60}$	$\frac{1}{60}$	$\frac{2}{60}$	$\frac{3}{60}$

Ex 5(例 2.3)　假設夫妻所得如下表：

夫	25	35	30	30	30	25	30	30	35	30
妻	20	25	25	20	20	25	30	25	30	30

1. Find 夫妻所得 joint 機率分配 & marginal pdf.
2. 分別計算夫妻所得期望值.
3. 求夫妻所得共變異數.
4. 求夫妻所得相關係數.
5. 列出妻所得為 20 時,夫所得之條件機率分配表,條件期望值.
6. 夫妻所得是否獨立？
7. 若政府對夫妻課固定比列稅率分別為 0.06 與 0.04 求夫妻稅後所得總額期望值.

$\Rightarrow$

1. 夫所得：25, 30, 35 (X)

妻所得：20, 25, 30 (Y)

$x \backslash y$	20	25	30	$f_X(x)$
25	$\frac{1}{10}$	$\frac{1}{10}$	0	$\frac{2}{10}$
30	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{6}{10}$
35	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$
$f_Y(y)$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	1

2. $E(X)=\sum_x x f_X(x)=25\cdot\frac{2}{10}+30\cdot\frac{6}{10}+35\cdot\frac{2}{10}=30$

$E(Y)=\sum_y y f_Y(y)=20\cdot\frac{3}{10}+25\cdot\frac{4}{10}+30\cdot\frac{3}{10}=25$

3. $Cov(X,Y)=E(XY)-E(X)E(Y)=755-30\times 25=5$

$$\because \begin{aligned}E(XY)&=25\cdot 20\cdot \tfrac{1}{10}+25\cdot 25\cdot \tfrac{1}{10}+30\cdot 20\cdot \tfrac{2}{10}+30\cdot 25\cdot \tfrac{2}{10}\\&+30\cdot 30\cdot \tfrac{2}{10}+35\cdot 25\cdot \tfrac{1}{10}+35\cdot 30\cdot \tfrac{1}{10}=755\end{aligned}$$

4. $Corr(X,Y)=\dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}=\dfrac{5}{\sqrt{10\times 15}}$

$$Var(X)=E(X^2)-E(X)^2=E(X^2)-(30)^2=10$$

$$E(X^2)=\sum_x x^2 f_X(x)=(25)^2\cdot \tfrac{2}{10}+(30)^2\cdot \tfrac{6}{10}+(35)^2\cdot \tfrac{2}{10}$$

同理 $Var(Y)=15$

5. $$P\left(X = x \middle| Y = 20\right) = \frac{P\left(X = x, Y = 20\right)}{P\left(Y = 20\right)} \qquad x = 25,30,35$$

$$= \frac{P\left(X = x, Y = 20\right)}{\frac{3}{10}} \qquad x = 25,30,35$$

$$= \begin{cases} \dfrac{\frac{1}{10}}{\frac{3}{10}} = \dfrac{1}{3} & x = 25 \\ \dfrac{\frac{2}{10}}{\frac{3}{10}} = \dfrac{2}{3} & x = 30 \\ 0 = 0 & x = 35 \end{cases}$$

x	25	30	35
$f_{X\|Y}\left(x\|20\right)$	$\frac{1}{3}$	$\frac{2}{3}$	0

$$E(X|Y=20)=\sum_x x f_{X|Y}(x|20)=25\cdot\tfrac{1}{3}+30\cdot\tfrac{2}{3}=\tfrac{85}{3}$$

6. $\because \quad f_{X,Y}(25,20)=\tfrac{1}{10}\neq\tfrac{3}{10}\cdot\tfrac{2}{10}=f_X(25)\cdot f_Y(20)$

$\therefore$ X 與 Y 不獨立

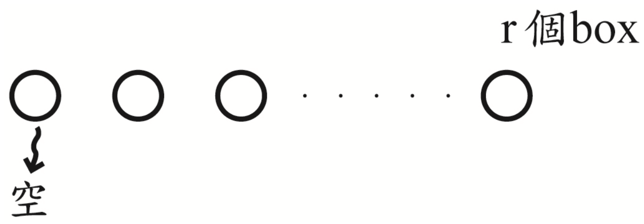
7. $Z=0.94X+0.96Y$

$$E(Z)=0.94E(X)+0.96E(Y)=52.2$$

Ex 6(例 2.4)　假設 n 個球隨機分布於 r 個盒子中.令 $X_i = 1$ 表示 i^{th} 盒子是空的, $X_i = 0$ 表示至少有一個球.

1. 計算 $E(X_i)$
2. $\forall i \neq j$ 計算 $E\left(X_i X_j\right)$
3. 令 S_r 表示空合數目, $S_r = X_1 + X_2 + \cdots + X_r$, 計算 $E(S_r)$
4. 計算 $Var(S_r)$

$\Rightarrow$



$$P(X_i = 1) = \frac{(r-1)^n}{r \cdot r \cdots r} = \frac{(r-1)^n}{r^n} = \left(1 - \tfrac{1}{r}\right)^n$$

1. $\therefore$

X_i	1	0
$P(X_i)$	$\left(1-\tfrac{1}{r}\right)^n$	$1-\left(1-\tfrac{1}{r}\right)^n$

$$E(X_i) = \sum_{x_i} x_i f_{X_i}(x_i) = \left(1-\tfrac{1}{r}\right)^n,$$

$$Var(X_i) = E(X_i^2) - E(X_i)^2 = \left(1-\tfrac{1}{r}\right)^n - \left(1-\tfrac{1}{r}\right)^{2n}$$

2.

x_i \ x_j	1	0
1	$\frac{(r-2)^n}{r^n}$	$\left(1-\tfrac{1}{r}\right)^n$
0	$\left(1-\tfrac{1}{r}\right)^n$	0

=

x_i \ x_j	1	0
1	$\left(1-\tfrac{2}{r}\right)^n$	$\left(1-\tfrac{1}{r}\right)^n$
0	$\left(1-\tfrac{1}{r}\right)^n$	0

$$\therefore \quad E\left(X_i X_j\right) = \left(1 - \tfrac{2}{r}\right)^n$$

3. $E\left(S_r\right) = E\left(X_1\right) + E\left(X_2\right) + \cdots + E\left(X_r\right) = r \cdot \left(1 - \tfrac{1}{r}\right)^n$

4.

$$Var\left(S_r\right) = Var\left(X_1 + X_2 + \cdots + X_r\right) = Cov\left(\sum_{i=1}^{r} X_i, \sum_{j=1}^{r} X_j\right) = \sum_{i=1}^{r}\sum_{j=1}^{r} Cov\left(X_i, X_j\right)$$

$$Cov\left(X_i, X_j\right) = E\left(X_i X_j\right) - E\left(X_i\right) E\left(X_j\right)$$

$$= \left(1 - \tfrac{2}{r}\right)^n - \left(1 - \tfrac{1}{r}\right)^n \cdot \left(1 - \tfrac{1}{r}\right)^n$$

$$= \left(1 - \tfrac{2}{r}\right)^n - \left(1 - \tfrac{1}{r}\right)^{2n}$$

$$\therefore \sum_{i=1}^{r}\sum_{j=1}^{r} Cov\left(X_i, X_j\right) = \text{variance} + \text{covariance}$$

$$= r(1-\tfrac{1}{r})^n \cdot (1-(1-\tfrac{1}{r})^n) + (r^2 - r)\left[\left(1-\tfrac{2}{r}\right)^n - \left(1-\tfrac{1}{r}\right)^{2n}\right]$$

Ex7 (例 2.5) Two *r.v.*s X&Y. $E(X^k)$ & $E(Y^k) \neq 0 \quad \forall k = 1,2,3,\cdots$

若$\left(\frac{X}{Y}\right)$與 Y 獨立. Show that $E\left(\frac{X}{Y}\right)^k = \frac{E(X^k)}{E(Y^k)}$

$\Rightarrow$

$$E\left[\left(\frac{X}{Y}\right)^k \cdot Y^k\right] = E\left(\left(\frac{X}{Y}\right)^k\right)E(Y^k)$$

$$\Rightarrow E(X^k) = E\left(\left(\frac{X}{Y}\right)^k\right)E(Y^k)$$

$$\Rightarrow E\left(\frac{X}{Y}\right)^k = \frac{E(X^k)}{E(Y^k)}$$

Ex 8(例 2.12)　盒中有 3 球，標號分別為 1,2,3. 以不歸還方式取 2 球.
若以 X,Y 分別表是第 1 球與第 2 球號碼.

1. 求 X,Y 聯合機率函數
2. 求 $Cov(X,Y)$ 與 $Corr(X,Y)$

$\Rightarrow$

1. $x=1,2,3$
   $y=1,2,3$

$x \backslash y$	1	2	3	$f_X(x)$
1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$
2	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{2}{6}$
3	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{2}{6}$
$f_Y(y)$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	1

2. $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \sum_x \sum_y xy f_{X,Y}(x,y) = 1 \cdot 2 \cdot \tfrac{1}{6} + 1 \cdot 3 \cdot \tfrac{1}{6} + 2 \cdot 2 \cdot \tfrac{1}{6} + 2 \cdot 3 \cdot \tfrac{1}{6}$$

$$+ 3 \cdot 1 \cdot \tfrac{1}{6} + 3 \cdot 2 \cdot \tfrac{1}{6} = \tfrac{11}{3}$$

$$E(X) = 1 \cdot \tfrac{2}{6} + 2 \cdot \tfrac{2}{6} + 3 \cdot \tfrac{2}{6} = 2$$

$$E(Y) = 1 \cdot \tfrac{2}{6} + 2 \cdot \tfrac{2}{6} + 3 \cdot \tfrac{2}{6} = 2$$

$$\therefore \quad Cov(X,Y) = \frac{11}{3} - 2^2 = -\frac{1}{3}$$

$$Var(X) = E(X^2) - E(X)^2 = 1^2 \cdot \frac{2}{6} + 2^2 \cdot \frac{2}{6} + 3^2 \cdot \frac{2}{6} - 2^2 = \frac{14}{3} - 4 = \frac{2}{3}$$

$$Var(Y) = \frac{2}{3}$$

$$Corr(X,Y) = \frac{-\frac{1}{3}}{\sqrt{\frac{2}{3} \cdot \frac{2}{3}}} = -\frac{1}{2}$$

Ex 9(例 2.24)　考慮以下聯合機率函數

$$f(x,y)=(0.6)^{x}(0.4)^{1-x}(0.3)^{y}(0.52)^{1-y}2^{xy} \quad x=0,1 \quad y=0,1$$

1. Find $f(y|x=0)$
2. $E(X)\&Var(X)$
3. $Cov(X,Y)$
4. $E(X+Y)$

$\Rightarrow$

1.

$x \backslash y$	0	1	$f_X(x)$
0	0.208	0.12	0.328
1	0.312	0.36	0.672
$f_Y(y)$	0.52	0.48	1

$$f_Y\left(y \middle| x=0\right)=\frac{f_{X,Y}(0,y)}{f_X(0)}=\frac{f_{X,Y}(0,y)}{0.328}=\begin{cases}\frac{0.208}{0.328}=\frac{26}{41} & y=0\\ \frac{0.12}{0.328}=\frac{15}{41} & y=1\end{cases}$$

2. $E(X)=\sum_x x f_X(x)=0.672$

$$E(X^2)=\sum_x x^2 f_X(x)=0.672$$

$$\therefore \quad Var(X)=0.672-0.672^2=0.2204$$

3. $Cov(X,Y) = E(XY) - E(X)E(Y) = 0.36 - (0.672)(0.48) = 0.03744$

$$E(Y) = \sum_y y f_Y(y) = 0.48$$

4. $E(X+Y) = E(X) + E(Y) = 0.672 + 0.48 = 1.152$

Ex 10(例 2.32) 隨機變數(X,Y),其 joint pdf 為 $f(X,Y)$. 假設 X,Y 的 mean 與 variance 分別為$\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2$且$\rho(X,Y)=\rho$.試以最小平方法求 a,b 值. s.t. $Q(a,b)=E\left[(Y-a-bX)^2\right]$ 為最小.以獲得 $\widehat{Y}=a+bX$.

$\Rightarrow$

$$\underset{(a,b)}{Min}\,Q(a,b)=E\left[(Y-a-bX)^2\right]$$

$$\begin{cases}\frac{\partial Q}{\partial a}=2E[(Y-a-bX)(-1)]=0\\ \frac{\partial Q}{\partial b}=2E[(Y-a-bX)(-X)]=0\end{cases}$$

$\Rightarrow$

$$\begin{cases}E(Y)=a+bE(X)\\ E(XY)=aE(X)+bE(X^2)\end{cases}$$

$\Rightarrow$

$$\begin{cases} a + E(X)b = E(Y) \\ E(X)a + E(X^2)b = E(XY) \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & E(X) \\ E(X) & E(X^2) \end{vmatrix} = E(X^2) - E(X)^2 = Var(X)$$

$$\Delta_b = \begin{vmatrix} 1 & E(Y) \\ E(X) & E(XY) \end{vmatrix} = E(XY) - E(X)E(Y) = Cov(X,Y)$$

$$\therefore \quad b = \frac{\Delta_b}{\Delta} = \frac{Cov(X,Y)}{Var(X)} = \frac{Cov(X,Y)\sqrt{Var(Y)}}{\sqrt{Var(X)Var(Y)}\sqrt{Var(X)}} = \rho \cdot \left(\frac{\sigma_Y}{\sigma_X} \right)$$

$$\because \quad a + b\mu_X = \mu_Y \quad \Rightarrow \quad a = \mu_Y - b\mu_X = \mu_Y - \rho\left(\frac{\sigma_Y}{\sigma_X} \right)\mu_X$$

$$\therefore \quad \widehat{Y} = \mu_Y - \rho\left(\frac{\sigma_Y}{\sigma_X}\right)\mu_X + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)X$$

$$= \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(X - \mu_X)$$