第五章 雙變量隨機變數

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第 5.6 節 連續隨機變數的動差

若 *r.vs.* X, Y ,其 joint *pdf* 為 $f_{X,Y}(x,y)$, 則

$$E(X)=\int_{-\infty}^{\infty} x f_X(x)dx=\int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\right]dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)dydx$$

同理

$$E(Y)=\int_{-\infty}^{\infty} y f_Y(y)dy=\int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\right]dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)dxdy$$

與 discrete *r.v.*相同，

$$Var(X)=E(X^2)-E(X)^2,\ Var(Y)=E(Y^2)-E(Y)^2$$

X , Y 的交叉動差：$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf_{X,Y}(x,y)dydx$

共變異數與相關係數 (與 discrete *r.v.*相同)

$$Cov(X,Y)=E(XY)-E(X)E(Y)$$

$$Corr(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

另外， $E(aX+bY)=aE(X)+bE(Y)$

$$Cov(aX+c,bY+d)=abCov(X,Y)$$

$$Var(aX+bY)=a^2Var(X)+b^2Var(Y)+2abCor(X,Y)$$

若 X 與 Y 獨立 $\Rightarrow$ $E(XY)=E(X)E(Y)$

$$\Rightarrow \quad Cov(X,Y)=0$$

Note： $Cov(X,Y)=0$, X,Y 未必 independent.

Ex 5.21 由 Ex 5.18 $f_X(x)=\begin{cases} \left(\frac{x+1}{4}\right) & , \quad 0\le x\le 2 \\ 0 & , \quad o.w. \end{cases}$

$$\therefore\ E(X)=\int_{-\infty}^{\infty} x f_X(x)dx=\int_0^2 x\left(\tfrac{x+1}{4}\right)dx=\tfrac{1}{4}\int_0^2\left(x^2+x\right)dx$$

$$=\tfrac{1}{4}\left(\tfrac{1}{3}x^3+\tfrac{1}{2}x^2\right)\Big|_0^2=\tfrac{1}{4}\left(\tfrac{8}{3}+\tfrac{4}{2}\right)=\tfrac{7}{6}$$

$$E\left(X^2\right)=\int_{-\infty}^{\infty} x^2 f_X(x)dx=\int_0^2 x^2\left(\tfrac{x+1}{4}\right)dx=\tfrac{1}{4}\int_0^2\left(x^3+x^2\right)dx$$

$$=\tfrac{1}{4}\left(\tfrac{1}{4}x^4+\tfrac{1}{3}x^3\right)\Big|_0^2=\tfrac{1}{4}\left(\tfrac{16}{4}+\tfrac{8}{3}\right)=\tfrac{5}{3}$$

$$\therefore\ Var(X)=\tfrac{5}{3}-\left(\tfrac{7}{6}\right)^2=\tfrac{11}{36}$$

同理, $E(Y)=\tfrac{7}{6}$ ，$Var(Y)=\tfrac{11}{36}$

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dydx=\int_0^2\int_0^2 xy\tfrac{1}{8}(x+y)dydx$$

$$=\tfrac{1}{8}\int_0^2\int_0^2 xy(x+y)dydx=\tfrac{1}{8}\int_0^2\int_0^2\left(x^2y+xy^2\right)dydx$$

$$=\tfrac{1}{8}\int_0^2\left[\tfrac{1}{2}x^2y^2+\tfrac{1}{3}xy^3\right]_{y=0}^{y=2}dx=\tfrac{1}{8}\int_0^2\left(2x^2+\tfrac{8}{3}x\right)dx$$

$$=\tfrac{1}{8}\left(\tfrac{2}{3}x^3+\tfrac{4}{3}x^2\right)\Big|_0^2=\tfrac{1}{8}\left(\tfrac{16}{3}+\tfrac{16}{3}\right)=\tfrac{4}{3}$$

$$\therefore\ \ Cov(X,Y)=E(XY)-E(X)E(Y)=\tfrac{4}{3}-\tfrac{7}{6}\cdot\tfrac{7}{6}=-\tfrac{1}{36}$$

$$Corr(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=\frac{-\tfrac{1}{36}}{\tfrac{11}{36}}=-\frac{1}{11}$$