第五章 雙變量隨機變數

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第 5.5 節　連續隨機變數的邊際分配

X,Y ~ Continuous r.v.s with joint pdf $f_{X,Y}(x,y)$. 則 X 的邊際機率密度函數 (marginal *pdf*) 為：

$$f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \qquad \text{(discrete } f_X(x)=\sum_y f_{X,Y}(x,y) \text{)}$$

同理 Y 的 marginal *pdf* 為：

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

邊際累積分配函數 (marginal *cdf*)：

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

(marginal *cdf* 是 marginal *pdf* 積分)

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt$$

Ex 5.17 假設 $f_{X,Y} = \begin{cases} \frac{1}{4} & , \quad 0 \le x \le 2, 0 \le y \le 2 \\ 0 & , \quad o.w. \end{cases}$

The marginal *pdf* of X is $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{4} dy = \frac{1}{4} y \Big|_0^2 = \frac{1}{2}$

$\therefore \quad f_X(x) = \begin{cases} \frac{1}{2} & , \quad 0 \le x \le 2 \\ 0 & , \quad o.w. \end{cases}$ i.e. $X \sim U(0,2)$

同理

$f_Y(y) = \begin{cases} \frac{1}{2} & , \quad 0 \le y \le 2 \\ 0 & , \quad o.w. \end{cases}$ i.e. $Y \sim U(0,2)$

The *cdf* of X is $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)dt$

$$= \begin{cases} if \quad x \le 0 \quad , & \int_{-\infty}^{x} 0dt = 0 \\ if \quad 0 \le x \le 2 \quad , & \int_{0}^{x} \frac{1}{2} dt = \frac{1}{2} x \\ if \quad x \ge 2 \quad , & 1 \end{cases}$$

$$\therefore \quad F_X(x) = \begin{cases} 0 & if \quad x \le 0 \\ \frac{x}{2} & if \quad 0 \le x \le 2 \\ 1 & if \quad x \ge 2 \end{cases}$$

Ex 5.18 $f_{X,Y}(x,y)=\begin{cases} \frac{1}{8}(x+y) & , \quad 0\le x\le 2, 0\le y\le 2 \\ 0 & , \quad o.w. \end{cases}$

$\Rightarrow$ The marginal *pdf* of X is $f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$$=\int_0^2 \tfrac{1}{8}(x+y)dy=\tfrac{1}{8}\int_0^2 (x+y)dy=\tfrac{1}{8}\left(xy+\frac{1}{2}y^2\right)\Bigg|_{y=0}^{y=2}$$

$$=\tfrac{1}{8}(2x+2)=\left(\tfrac{x+1}{4}\right) \qquad 0\le x\le 2$$

$$\therefore \quad f_X(x)=\begin{cases} \left(\tfrac{x+1}{4}\right) & , \quad 0\le x\le 2 \\ 0 & , \quad o.w. \end{cases}$$

The *cdf* of X is

$$\forall x \in \Re,\quad F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & if \quad x \le 0 \\ \int_0^x \left(\frac{t+1}{4}\right) dt & if \quad 0 < x \le 2 \\ 1 & if \quad x > 2 \end{cases}$$

$$\int_0^x \left(\tfrac{t+1}{4}\right) dt = \tfrac{1}{4}\int_0^x (t+1)\,dt = \tfrac{1}{4}\left[\tfrac{1}{2}t^2 + t\right]_0^x$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}x^2 + x\right) = \tfrac{1}{8}x^2 + \tfrac{1}{4}x$$

$$\therefore F_X(x) = \begin{cases} 0 & if \quad x \le 0 \\ \tfrac{1}{8}x^2 + \tfrac{1}{4}x & if \quad 0 \le x \le 2 \\ 1 & if \quad x \ge 2 \end{cases}$$

***Note*：**

If X, Y independent (與 discrete 相同)

- $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \quad \forall x, y \in \Re$
- $F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y) \quad \forall x, y \in \Re$

Ex 5.19 根據 Ex 5.15, $f_{X,Y}(x,y)=\begin{cases}\frac{1}{4} & , \quad 0\le x\le 2,\ 0\le y\le 2\\ 0 & , \quad o.w.\end{cases}$

$\because f_X(x)=\begin{cases}\frac{1}{2} & , \quad 0\le x\le 2\\ 0 & , \quad o.w.\end{cases}$ $f_Y(y)=\begin{cases}\frac{1}{2} & , \quad 0\le y\le 2\\ 0 & , \quad o.w.\end{cases}$

$\therefore \quad f_X(x,y)=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{2}=f_X(x)\cdot f_Y(y) \quad \forall x,y\in\Re$

$\therefore$ X 與 Y 獨立

根據 Ex 5.16 $f_{X,Y}(x,y)=\begin{cases} \frac{1}{8}(x+y) & , \quad 0\le x\le 2, 0\le y\le 2 \\ 0 & , \quad o.w. \end{cases}$

$$\Rightarrow \quad f_X(x)=\begin{cases} \frac{x+1}{4} & , \quad 0\le x\le 2 \\ 0 & , \quad o.w. \end{cases}$$

$$f_Y(x)=\begin{cases} \frac{y+1}{4} & , \quad 0\le y\le 2 \\ 0 & , \quad o.w. \end{cases}$$

$$\Rightarrow \quad f_{X,Y}(x,y)\neq f_X(x)\cdot f_Y(y)$$

$\therefore$ X 與 Y 相依