# 第五章 雙變量隨機變數

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# 第 5.4 節 連續隨機變數的聯合分配

***Recall*：** X 與 Y 的 joint cdf $F_{X,Y}$ 定義為 $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

若 $F_{X,Y}$ 為連續函數，且除了有限個點以外均為可微，則稱 X,Y 具有連續的聯合分配 (continuous joint distribution)

令 $f_{X,Y}(x,y) = \dfrac{\partial F_{X,Y}(x,y)}{\partial x \partial y}$

$f_{X,Y}$ 稱為聯合機率密度函數( joint *pdf* )，其必定滿足：

- $f_{X,Y}(x,y) \geq 0$ ，$\forall\ x,y \in \Re$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = 1$

連續型 joint *cdf* 為

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)dsdt \qquad \text{(雙重積分)}$$

**比較：** discrete joint *cdf*: $F_{X,Y}(x,y) = \sum_{s \leq x}\sum_{t \leq y} f_{X,Y}(s,t)$

**Ex 5.15** X,Y 在$[r,s]\times[u,v]$的 joint *pdf* 為

$$f_{X,Y}(x,y)=\begin{cases} c\ , & r\le x\le s\ ,\ u\le y\le v \\ 0\ , & o.w. \end{cases}$$

$\Rightarrow$

$\because\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx=1$

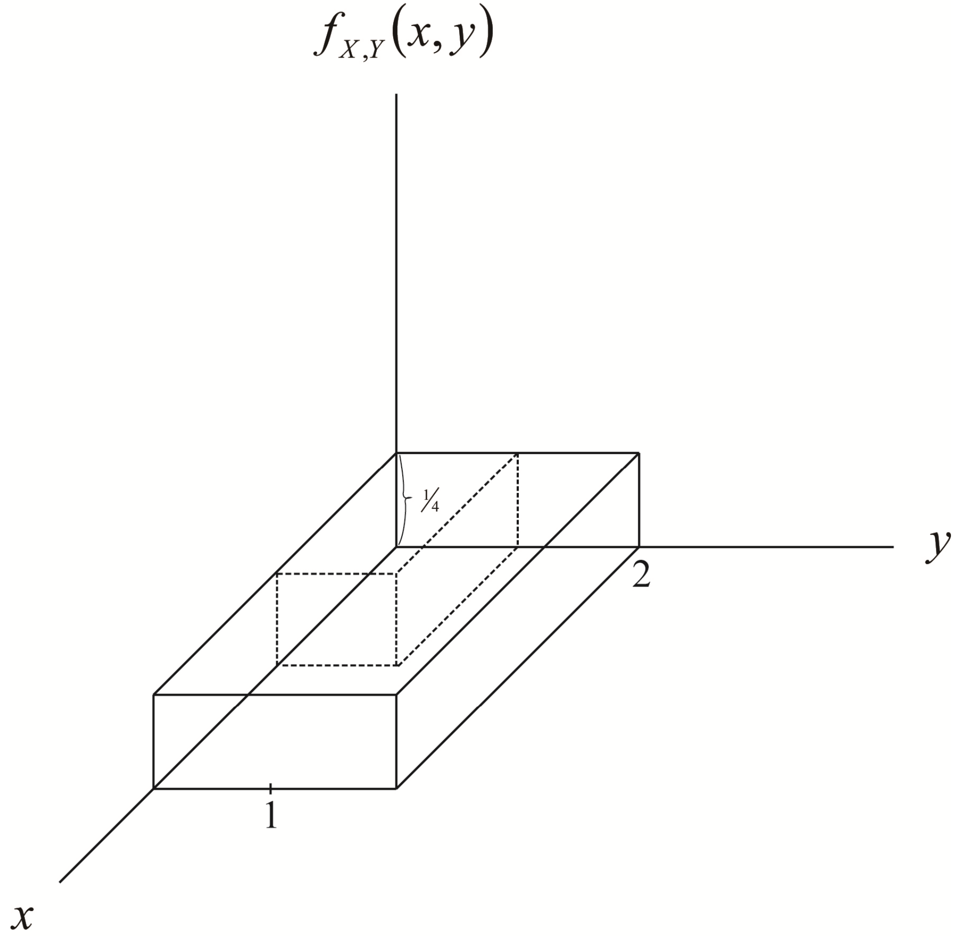
$\Rightarrow\quad \int_r^s\int_u^v cdydx=c(s-r)(v-u)=1$

$\Rightarrow\quad c=\dfrac{1}{(s-r)(v-u)}$

例如：$[r,s]$, $[u,v]$皆為$[0,2]$.

則：

$$f_{X,Y}(x,y)=\begin{cases} 1/4\ , & 0\le x\le 2 \\ 0\ , & 0\le y\le 2 \end{cases}$$



另外,

$$F_{X,Y}(1,1)=\int_{-\infty}^{1}\int_{-\infty}^{1}\tfrac{1}{4}\,dy\,dx=\int_{0}^{1}\int_{0}^{1}\tfrac{1}{4}\,dy\,dx=\tfrac{1}{4}$$

$$F_{X,Y}(1,1.5)=\int_{-\infty}^{1}\int_{-\infty}^{1.5}\tfrac{1}{4}\,dy\,dx=\int_{0}^{1}\int_{0}^{1.5}\tfrac{1}{4}\,dy\,dx=\tfrac{3}{8}$$

$$F_{X,Y}(1.5,1)=\tfrac{3}{8}$$

***Note***： $F_{X,Y}$具有非遞減性，無論 X,Y 為 discrete or continuous.

**Ex 5.16** 考慮以下 joint *pdf* $f_{X,Y}(x,y)=\begin{cases} h(x+y), & 0\le x\le 2 \\ & 0\le y\le 2 \\ 0, & o.\text{w.} \end{cases}$

$$\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx=1$$

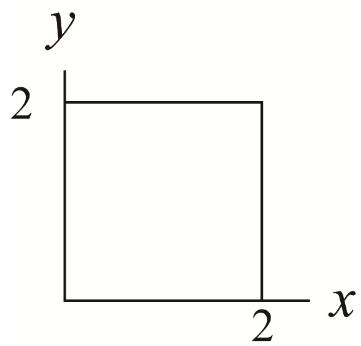
$$\Rightarrow \quad \int_0^2\int_0^2 h(x+y)dydx=1$$

$$=h\int_0^2\int_0^2 (x+y)dydx=h\int_0^2\left(xy+\tfrac{1}{2}y^2\right)\Big|_{y=0}^{2}dx$$

$$=h\int_0^2(2x+2)dx=h\cdot\left(x^2+2x\right)\Big|_0^2=h(4+4)=8h$$

$$\therefore \; 8h=1 \quad \Rightarrow \quad h=\tfrac{1}{8}$$

$$\therefore \quad f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(x+y) \ , & 0 \le x \le 2 \\ & 0 \le y \le 2 \\ 0 \ , & o.\text{w.} \end{cases}$$




The joint *cdf* of $(x,y)$ is

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)dsdt$$

$$= \begin{cases} \int_0^x\int_0^y \frac{1}{8}(s+t)dsdt \ , & 0 \le x \le 2 \\ & 0 \le y \le 2 \\ 1 \ , & x > 2 \\ & y > 2 \\ 0 \ , & o.\text{w.} \end{cases}$$

$$\frac{1}{8}\int_0^x \left(\frac{1}{2}s^2 + ts\right)\Big|_{s=0}^{s=y} dt = \frac{1}{8}\int_0^x \left(\frac{1}{2}y^2 + ty\right) dt = \frac{1}{8}\left(\frac{y^2}{2}t + \frac{y}{2}t^2\right)\Big|_{t=0}^{t=x}$$

$$= \frac{1}{8}\left(\frac{y^2}{2}x + \frac{y}{2}x^2\right) = \frac{1}{16}xy(x+y)$$

$$\therefore \quad F_{X,Y}(x,y) = \begin{cases} \frac{1}{16}xy(x+y), & 0 \le x \le 2 \\ & 0 \le y \le 2 \\ 1, & x > 2 \\ & y > 2 \\ 0, & o.w. \end{cases}$$

***Note***：

$X, Y$ continuous *r.v.*s with joint *pdf* $f_{X,Y}(x,y)$

$$\Rightarrow \quad P(a_1 \le X \le a_2, b_1 \le Y \le b_2) = \int_{a_1}^{a_2}\left(\int_{b_1}^{b_2} f_{X,Y}(x,y)\,dy\right)dx$$

i.e. $f_{X,Y}(x,y)$ 曲面下與 $a_1 \le x \le a_2$ ， $b_1 \le y \le b_2$ 所包圍的體積。