第四章 機率基礎概念

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第 4.5 節 連續隨機變數的動差與分位數

4.5-1 均數與變異數

X 為 continuous *r.v.*,其 *pdf* 為 $f_X(x)$, 則 X 的期望值與變異數分別定義如下：

$$E(X)=\int_{-\infty}^{\infty} x f_X(x)dx \qquad \text{其中 } E(X)\equiv\mu$$

$$Var(X)=\int_{-\infty}^{\infty}(x-\mu)^2 f_X(x)dx$$

***Remarks*：**

1. $E(aX+b)=aE(X)+b$

2. $Var(aX+b)=a^2Var(X)$

3. $E(g(X))=\int_{-\infty}^{\infty} g(x)f_X(x)dx$

4. $Var(g(X))=\int_{-\infty}^{\infty} (g(x)-E(g(x)))^2 f_X(x)dx$

Ex 4.13 $X \sim U(r,s),\quad E(X)=?\ ,\quad Var(X)=?$

$\Rightarrow$

$$E(X)=\int_{-\infty}^{\infty} x f_X(x)dx$$

$$=\int_{-\infty}^{r} x\cdot 0dx+\int_{r}^{s} x\cdot \frac{1}{s-r}dx+\int_{s}^{\infty} x\cdot 0dx$$

$$=\frac{1}{s-r}\int_{r}^{s} xdx=\frac{1}{s-r}\cdot\frac{1}{2}x^2\Big|_r^s=\frac{1}{2}\cdot\frac{1}{s-r}\left(s^2-r^2\right)$$

$$=\frac{1}{2}\frac{1}{s-r}(s+r)(s-r)=\frac{s+r}{2}\equiv\mu$$

$$Var(X) = \int_{-\infty}^{0} (x-\mu)^2 f_X(x)dx$$

$$= \int_{-\infty}^{r} (x-\mu)^2 \cdot 0dx + \int_{r}^{s} (x-\mu)^2 \cdot \frac{1}{s-r}dx + \int_{s}^{\infty} (x-\mu)^2 \cdot 0dx$$

$$= \frac{1}{s-r} \cdot \frac{1}{3}(x-\mu)^3 \Big|_{x=r}^{x=s} = \frac{1}{3} \cdot \frac{1}{s-r}\left[(s-\mu)^3 - (r-\mu)^3\right]$$

$$= \frac{1}{3} \cdot \frac{1}{s-r}\left[\left(\frac{s-r}{2}\right)^3 - \left(\frac{r-s}{2}\right)^3\right]$$

$$= \frac{1}{3} \cdot \frac{1}{s-r} \cdot \left(\frac{s-r}{2}\right)^3 \times 2 = \frac{1}{12}(s-r)^2$$

***Note*：** *if* $X \sim U(0,1) \Rightarrow E(X) = \frac{1}{2}$, $Var(X) = \frac{1}{12}$

4.5-2　其他指標

X 為 *r.v.*, F_X 為其 *cdf*。令 $o<q<1$，F_X 的第 q 分位數(quantile)為：

滿足 $F_X(x)\geq q$ 中，最小的 x 值，以 x_q 表示。

i.e. $x_q = Min\{x|F_X(x)\geq q\}$

Ex $q=0.5,\quad x_{0.5}=Min\{x|F_X(x)\geq 0.5\}$

此時 $\begin{cases} X > x_{0.5}\text{的機率為}0.5 \\ X < x_{0.5}\text{的機率為}0.5 \end{cases}$　　*i.e.* $x_{0.5}$ 為中位數

四分位數為　$x_{0.25}$，　$x_{0.5}$，　$x_{0.75}$

百分位數為　$x_{0.01}$，　$x_{0.02}, \cdots$　$x_{0.99}$

$\because$　$x_{0.5}$　為中位數

$$\therefore \; P\left(X \le x_{0.5}\right) = \int_{-\infty}^{x_{0.5}} f_X\left(x\right)dx = 0.5$$

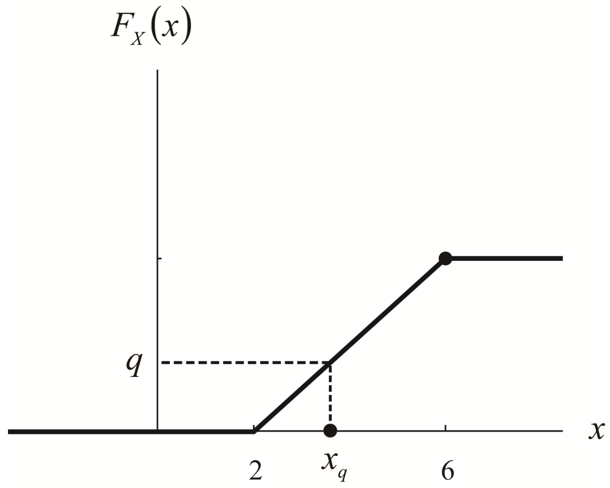
$$P\left(X \ge x_{0.5}\right) = \int_{x_{0.5}}^{\infty} f_X\left(x\right)dx = 0.5$$

***Note*：**　眾數為　f_X　中最高點所對應之　x　值。

一個分配可能有很多眾數，也可能沒有。

Ex 4.14 $X \sim U(2,6)$

$$\Rightarrow \quad F_X(x)=\begin{cases} 0 & x\le 2 \\ \dfrac{x-2}{4} & 2<x\le 6 \\ 1 & x>6 \end{cases}$$



給定 q 值 $o<q<1$

$\because \; x_q = Min\{x|F_X(x)\ge q\}$

$$F_X(x)\ge q \;\Rightarrow\; \frac{x-2}{4}\ge q \;\Rightarrow\; x-2\ge 4q \;\Rightarrow\; x\ge 4q+2$$

$$Min\{x|F_X(x)\ge 4q+2\}=x_q=4q+2$$

Ex $q = 0.25$, $x_{0.25} = 2 + 4\left(1/4\right) = 3$

$q = 0.5$, $x_{0.5} = 2 + 4\left(1/2\right) = 4$

定理 4.15（才比雪夫不等式 Chebyshev inequatity）

X is a *r.v.* with finite variance, $Var(X)$, the $\forall\ c > 0$, we have

$$P\left(\left|X - E(X)\right| \geq c\right) \leq \frac{Var(X)}{c^2}$$

若取 $c = k\sigma_X$ ，則

$$P\left(\left|X - \mu\right| \geq k\sigma_X\right) \leq \frac{\sigma_X^2}{k^2\sigma_X^2} = \frac{1}{k^2} \quad \text{or} \quad P\left(\left|X - \mu\right| < k\sigma_X\right) \geq 1 - \frac{1}{k^2}$$

Proof:

假設 X 為 continuous r.v.

$$\because \quad Var(X)=\int_{-\infty}^{\infty}(x-\mu)^2 f_X(x)dx$$

$$=\int_{-\infty}^{\mu-k\sigma_X}(x-\mu)^2 f_X(x)dx+\int_{\mu-k\sigma_X}^{\mu+k\sigma_X}(x-\mu)^2 f_X(x)dx+\int_{\mu+k\sigma_X}^{\infty}(x-\mu)^2 f_X(x)dx$$

$$\Rightarrow \sigma_X^2 \ge \int_{-\infty}^{\mu-k\sigma_X}(x-\mu)^2 f_X(x)dx+\int_{\mu+k\sigma_X}^{\infty}(x-\mu)^2 f_X(x)dx$$

$x \le \mu - k\sigma_X$ $\qquad\qquad$ $x \ge \mu + k\sigma_X$

$$\Rightarrow (x-\mu) \le -k\sigma_X \qquad\qquad \Rightarrow (x-\mu) \ge k\sigma_X$$

$$\Rightarrow (x-\mu)^2 \ge k^2\sigma_X^2 \qquad\qquad \Rightarrow (x-\mu)^2 \ge k^2\sigma_X^2$$

$$\Rightarrow \sigma_X^2 \ge \int_{-\infty}^{\mu-k\sigma_X} k^2\sigma_X^2 \, f_X(x)dx + \int_{\mu+k\sigma_X}^{\infty} k^2\sigma_X^2 \, f_X(x)dx$$

$$= k^2\sigma_X^2 \cdot P(|X-\mu| \ge k\sigma_X)$$

$$\Rightarrow \sigma_X^2 \ge k^2\sigma_X^2 \cdot P(|X-\mu| \ge k\sigma_X)$$

$$\Rightarrow P(|X-\mu| \ge k\sigma_X) \le \frac{1}{k^2}$$

補充：

動差與動差母函數(moment-generating function)

定義 4.1 r.v. Y, Y 的相對於原點 (origin) 的 k^{th} 動差為 $E\left(Y^k\right)$，以 μ_k' 表示。

Ex $\mu_1' = E(Y) = \mu$

$$\mu_2' = E\left(Y^2\right)$$

$$\sigma^2 = E\left(Y^2\right) - E(Y)^2 = \mu_2' - \left(\mu_1'\right)^2$$

定義 4.2 r.v. Y, Y 相對於 mean 的 k^{th} 中央動差(central moment) 為 $E\left[(Y-\mu)^k\right]$, 以 μ_k 表示。

Ex $\mu_2 = E\left[(Y-\mu)^2\right] = \sigma^2$

定義 4.3 The moment generating function (mgf) of a r.v. Y is defined to be $m(t) = E\left(e^{tY}\right)$。We say a *mgf* for Y exists if $\exists\ b > 0$ s.t. $m(t) < \infty$ $\forall\ |t| < b$

Recall： $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$\therefore \quad e^{tY} = 1 + tY + \frac{(tY)^2}{2!} + \frac{(tY)^3}{3!} + \frac{(tY)^4}{4!} + \cdots$

$$\Rightarrow E\left(e^{tY}\right) = \sum_y e^{ty} f_Y(y)$$

$$= \sum_y \left[1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \frac{(ty)^4}{4!} + \cdots \right] f_Y(y)$$

$$= \sum_y f_Y(y) + \sum_y ty f_Y(y) + \frac{1}{2!} \sum_y t^2 y^2 f_Y(y) + \frac{1}{3!} \sum_y t^3 y^3 f_Y(y) + \cdots$$

$$= 1 + tE(Y) + \frac{t^2}{2!} E\left(Y^2\right) + \frac{t^3}{3!} E\left(Y^3\right) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots \qquad (1)$$

定理 If $m(t)$ exist, then for any positive integer k,

$$\left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = m^{(k)}(0) = \mu_k'$$

proof：

由(1)

$$\Rightarrow m_Y'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3}{3!}t^2\mu_3' + \cdots \quad \Rightarrow \quad m_Y'(0) = \mu_1'$$

$$m_Y^{(2)}(t) = \mu_2' + \frac{2\times 3}{3!}t\mu_3' + \cdots \quad \Rightarrow \quad m_Y^{(2)}(0) = \mu_2'$$

$$\vdots$$

$$m_Y^{(k)}(0) = \mu_k'$$

Remark： *mgf* 唯一決定一種分配， *i.e.*

⇔If $m(t)$ exists for a probability function , it is unique.

⇔If *r.v.*'s Y & Z have the same *mgf*s , they have the same probability distribution .

Probability generating function (機率母函數)

定義 4.4　Y 是一個整數值的 *r.v.*, 且 $P(Y=i)=P_i$, $i=0,1,2,\cdots$, 則 Y 的機率母函數 (*pgf*) 定義為

$$P(t)=E\left(t^{Y}\right)=P_0+P_1t+P_2t^2+\cdots+\sum_{i=0}^{\infty}P_it^i$$

$$(\because\ P(t)=E\left(t^{Y}\right)=\sum_{Y}t^{Y}f_Y(Y)=\sum_{i=0}^{\infty}t^{i}P(Y=i)=\sum_{i=0}^{\infty}t^{i}P_i\ )$$

定義 4.5 The k^{th} factorial moment of a *r.v.* Y is defined to be

$$\mu_{[k]} = E[Y(Y-1)(Y-2)\cdots(Y-k+1)]$$

定理：If $P(t)$ is the *pgf* for an integer-valued *r.v.* Y, the k^{th} factorial moment of Y is

$$\left.\frac{d^k P(t)}{dt^k}\right|_{t=1} = P^{(k)}(1) = \mu_{[k]}$$

proof：

$$\because \; P(t) = E\left(t^Y\right) = P_0 + P_1 t + P_2 t^2 + P_3 t^3 + P_4 t^4 + \cdots + P_k t^k + \ldots$$

$$\Rightarrow P'(t) = P_1 + 2P_2 t + 3P_3 t^2 + 4P_4 t^3 + \cdots + kP_k t^{k-1} + \ldots$$

$$P''(t) = 2P_2 + 3 \cdot 2P_3 t^1 + 4 \cdot 3P_4 t^2 + \cdots + k \cdot (k-1) P_k t^{k-2}$$

$$\vdots$$

$$P^{(k)}(t) = \sum_{y=k}^{\infty} y \cdot (y-1)(y-2)\cdots(y-k+1) P_k t^{y-k}$$

$$\Rightarrow P'(1) = 1 \cdot P_1 + 2P_2 + 3P_3 + 4P_4 + \cdots = E(Y) = \mu_{[1]}$$

$$P''(1) = 2P_2 + 3 \cdot 2P_3 + 4 \cdot 3P_4 + \cdots = \mu_{[2]} = E[Y(Y-1)]$$

$$\vdots$$

$$P^{(k)}(1) = \sum_{y=k}^{\infty} y \cdot (y-1)(y-2)\cdots(y-k+1) P_k$$

$$= E[Y(Y-1)(Y-2)\cdots(Y-k+1)] = \mu_{[k]}$$

更多例子：

Ex 1(例 1.2) Consider the *r.v.* X with the probability distribution

x	-4	0	1	2
$P(x)$	0.2	0.3	0.4	0.1

1. $P(x>0)$ 2. $P(x \ge 0)$ 3. $P(0 \le x \le 1)$

4. $P(x=-4)$ 5. $P(x=-2)$

$\Rightarrow$

1. $P(x>0)=P(x=1)+P(x=2)=0.4+0.1=0.5$

2. $P(x \ge 0)=1-P(x<0)=1-P(x=-4)=1-0.2=0.8$

3. $P(0 \le x \le 1) = P(x = 0) + P(x = 1) = 0.3 + 0.4 = 0.7$

4. $P(x = -4) = 0.2$

5. $P(x = -2) = 0$

Ex 2(例 2.1) *A r.v. X* has the following probability function

x	0	1	2	3
$f(x)$	1/2	1/4	1/8	1/8

Let $Y = X^2 + 2X + 1$

1. Find $f_Y(y)$ 2. $E(Y) = ?$ 3. $Var(Y) = ?$

$\Rightarrow$

1.

y	1	4	9	16
$f(y)$	1/2	1/4	1/8	1/8

2. $E(Y) = \sum_y y f_Y(y) = 1\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 9\left(\frac{1}{8}\right) + 16\left(\frac{1}{8}\right) = 4.625$

3. $Var(Y) = E(Y^2) - (E(Y))^2$

$$= E(Y^2) - (4.625)^2 = 46.625 - (4.625)^2$$

$$\left(^{**} E(y^2) = \sum_y y^2 f_Y(y) = 1^2\left(\tfrac{1}{2}\right) + 4^2\left(\tfrac{1}{4}\right) + 9^2\left(\tfrac{1}{8}\right) + 16^2\left(\tfrac{1}{8}\right) = 46.625\right)$$

Ex 3(例 2.2) 一個 box 中，1 號球 1 個，2 號球 2 個，3 號球 3 個，⋯ n 號球 n 個，隨機抽取一球，以 *X* 表示抽出之號碼。

求 1. *X* 的機率分配 2. *E(X)*

$\Rightarrow$

1. 共有$1+2+\cdots+n=\dfrac{n(n+1)}{2}$個球

$$P(X=x)=\begin{cases}\dfrac{x}{\frac{n(n+1)}{2}} & , \ x=1,2,3\cdots n \\ 0 & , \ o.w.\end{cases}$$

$$=\begin{cases}\dfrac{2x}{n(n+1)} & , \ x=1,2,3\cdots n \\ 0 & , \ o.w.\end{cases}$$

2. $$E(X)=\sum_{x} x f_X(x)=\sum_{x=1}^{n} x\cdot\frac{2x}{n(n+1)}=\frac{2}{n(n+1)}\sum_{x=1}^{n} x^2$$

$$=\frac{2}{n(n+1)}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{2n+1}{3}$$

Ex 4(例 2.9) Show that $E\left(X^2\right) \geq (E(X))^2$. In what condition $E\left(X^2\right) = (E(X))^2$?

$\Rightarrow$

$\because \; Var(X) = E\left(X^2\right) - \left(E(X)\right)^2 \geq 0$

$\Rightarrow \quad E\left(X^2\right) \geq (E(X))^2$

If $Var(X) = 0 \Leftrightarrow E\left(X^2\right) = \left(E(X)\right)^2$

Ex 5(例 2.12) Suppose the *pf* of X is $f_X(x)=\dfrac{c}{x(x+1)}$ $x=2,3,\cdots$

1. Find $c=?$ 2. $E(X)=?$

$\Rightarrow$

1. $\because$ $\sum_x f_X(x)=1$

$$\Rightarrow \sum_x \frac{c}{x(x+1)}=1 \qquad \Rightarrow \quad c\cdot\sum_x \frac{1}{x(x+1)}=1$$

$$\Rightarrow c\cdot\sum_{x=2}^{\infty}\frac{1}{x(x+1)}=1 \qquad \Rightarrow \quad c\cdot\sum_{x=2}^{\infty}\left[\frac{1}{x}-\frac{1}{x+1}\right]=1$$

$$\Rightarrow c\cdot\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots\right]=1$$

$$\Rightarrow c=2$$

2. $$E(X) = \sum_{x} x f_X(x) = \sum_{x=2}^{\infty} x \cdot \frac{2}{x(x+1)}$$

$$= 2 \cdot \sum_{x=2}^{\infty} x \cdot \frac{1}{x(x+1)}$$

$$= \infty$$

Ex 6(例 2.27) A *r.v.* X has the following *cdf*

$$F_X(x)=\begin{cases} 0 & , \quad if \quad x<-1 \\ \frac{1}{3} & , \quad if \quad -1\le x<0 \\ \frac{2}{3} & , \quad if \quad 0\le x<1 \\ 1 & , \quad if \quad x\ge 1 \end{cases}$$

Find $E(X)$ and $Var(X)$

$\Rightarrow$

$\because \; E(X)=\sum_x x f_X(x)$

Must find $f_X(x)$ first

$x=-1,0,1$

$$f_X(-1) = F_X(-1) - F_X(-1^-) = \frac{1}{3} - 0 = \frac{1}{3}$$

$$f_X(0) = F_X(0) - F_X(0^-) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$f_X(1) = F_X(1) - F_X(1^-) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore f_X(x) = \begin{cases} \frac{1}{3} & , if \quad x = -1, 0, 1 \\ 0 & , if \quad o.w. \end{cases}$$

$$E(X) = \sum_x x f_X(x) = -1 \cdot \left(\frac{1}{3}\right) + 0 \cdot \left(\frac{1}{3}\right) + 1 \cdot \left(\frac{1}{3}\right) = 0$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = (-1)^2 \cdot \frac{1}{3} + 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} - 0 = \frac{2}{3}$$

Ex 7(例 3.1) A *r.v.* X has following probability function $f(x)$

x	$f(x)$
-1	0.5
0	0.3
1	0.1
3	0.1

1. Find the *cdf* of X
2. Find the *mgf* of X

$\Rightarrow$

1. The *cdf* of X is $F_X(x) = P(X \le x) \quad \forall x \in \Re$

$$= \begin{cases} 0 & if & x < -1 \\ 0.5 & if & -1 \le x < 0 \\ 0.8 & if & 0 \le x < 1 \\ 0.9 & if & 1 \le x < 3 \\ 1 & if & x \ge 3 \end{cases}$$

2. The *mdf* of X is $M_X(t) = E\left(e^{tx}\right) = \sum_x e^{tx} f(x)$

$$= e^{t(-1)} \times 0.5 + e^{t(0)} \times 0.3 + e^{t(1)} \times 0.1 + e^{t(3)} \times 0.1$$

$$= 0.5e^{-t} + 0.3 + 0.1e^{t} + 0.1e^{3t}$$

$$= 0.3 + 0.5e^{-t} + 0.1e^{t} + 0.1e^{3t}$$

Ex 8(例 3.3) The *mgf* of X is $M(t)$. Let $Y = aX + b$

1. Find the *mgf* of Y
2. Use the result in (1) to show $E(Y) = aE(X) + b$

$\Rightarrow$

1. The mgf of Y is $M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(aX+b)}\right) = E\left(e^{taX} \cdot e^{tb}\right)$

$$= e^{tb} \cdot E\left(e^{t(aX)}\right) = e^{tb} \cdot E\left(e^{atX}\right)$$

$$= e^{tb} \cdot M(at)$$

2. $M_Y{}'(t) = e^{tb} \cdot M'(at) \cdot a + M(at) \cdot e^{tb} \cdot b$

$$E(Y) = M_Y{}'(0) = M'(0) \cdot a + b = aE(X) + b$$

Ex 9(例 3.4) 1.假設$E(X^r)=0.8$, $r=1,2,\cdots$. Find the *mgf* of X.

2.假設$E(X^r)=5^r$, $r=1,2,\cdots$. Find the *mgf* of X.

$\Rightarrow$

1. The *mgf* of X is $M_X(t)=E(e^{tX})=1+\mu_1'\cdot\frac{t}{1!}+\mu_2'\cdot\frac{t^2}{2!}+\cdots$

$$\therefore M_X(t)=1+0.8t+0.8\cdot\frac{t^2}{2!}+\cdots$$

$$=1+0.8\sum_{k=1}^{\infty}\frac{t^k}{k!}=0.2+0.8+0.8\sum_{k=1}^{\infty}\frac{t^k}{k!}=0.2+0.8\sum_{k=0}^{\infty}\frac{t^k}{k!}$$

$$=0.2+0.8e^t$$

(***Note*:** $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$)

2. The *mgf* of X is $M_X(t) = 1 + \mu_1' \cdot \frac{t}{1!} + \mu_2' \cdot \frac{t^2}{2!} + \cdots$

$$= 1 + \sum_{k=1}^{\infty} \mu_k' \frac{t^k}{k!} = 1 + \sum_{k=1}^{\infty} 5^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} 5^k \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(5t)^k}{k!}$$

$$= e^{5t}$$

Ex 10(例 3.8) Let $M_X(t)$ denote the *mgf* of X . Show that

$$M_{\left(\frac{X-\mu}{\sigma}\right)}(t) = e^{-\frac{X-\mu}{\sigma}} \cdot M_X\left(\tfrac{t}{\sigma}\right)$$

$\Rightarrow$

Let $Y = \left(\dfrac{X-\mu}{\sigma}\right)$

the *mgf* of Y is $M_Y(t) = M_{\left(\frac{X-\mu}{\sigma}\right)}(t) = E\left(e^{tY}\right) = E\left(e^{t\left(\frac{X-\mu}{\sigma}\right)}\right)$

$= E\left(e^{t\left(\frac{X}{\sigma}\right) - \frac{t\mu}{\sigma}}\right) = E\left(e^{\left(\frac{t}{\sigma}\right)X} e^{-\frac{t\mu}{\sigma}}\right) = e^{-\frac{t\mu}{\sigma}} E\left(e^{\left(\frac{t}{\sigma}\right)X}\right)$

$= e^{-\frac{\mu}{\sigma}t} M_X\left(\tfrac{t}{\sigma}\right)$

Ex 11(例 3.9) The *mgf* of Y is $M(t) = (1-t)^{-2} \quad , t < 1$

Find $E(X)$ and σ^2

$\Rightarrow$

$$\because \quad E(X) = M'(0) = -2(1-t)^{-3}(-1)\Big|_{t=0} = 2$$

$$E(X^2) = M''(0) = 2 \cdot (-3)(1-t)^{-4}(-1)\Big|_{t=0} = 6$$

$$\sigma^2 = E(X^2) - (E(X))^2 = 6 - 2^2 = 2$$

連續隨機變數的例子

Ex 1(例 2.3) A *r.v.* X has the *pdf*

$$f(x)=\begin{cases} \frac{2x}{k^2} & , \quad 0 \le x \le k \\ 0 & , \quad o.w. \end{cases}$$

k 值為何 $Var(X)=2$

$\Rightarrow$

$$Var(X)=E(X^2)-(E(X))^2$$

$$E(X)=\int_{-\infty}^{\infty} xf(x)dx=\int_0^k x\frac{2x}{k^2}dx=\frac{2}{k^2}\int_0^k x^2dx=\frac{2}{k^2}\frac{1}{3}x^3\Big|_0^k=\frac{2}{3}k$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_0^k x^2\frac{2x}{k^2}dx=\frac{2}{k^2}\int_0^k x^3dx=\frac{2}{k^2}\frac{1}{4}x^4\Big|_0^k=\frac{1}{2k^2}k^4=\left(\frac{1}{2}\right)k^2$$

$$\therefore \; Var(x) = \tfrac{1}{2}k^2 - \left(\tfrac{2}{3}k\right)^2 = 2$$

$$\Rightarrow \; \tfrac{1}{2}k^2 - \tfrac{4}{9}k^2 = 2 \; \Rightarrow \; 9k^2 - 8k^2 = 36 \; \Rightarrow \; k^2 = 36$$

$\Rightarrow \; k^2 = \pm 6$ (負不合)

Ex2 (例 2.7) *r.v.* X with the *pdf* $f(x)=\begin{cases} \frac{x}{18} & , \quad 0\le x\le 6 \\ 0 & , \quad o.w. \end{cases}$

1. $E(X^3)$ 2. Median of X

$\Rightarrow$

1. $E(X^3)=\int_0^6 xf(x)dx=\int_0^6 x\frac{x}{18}dx=\frac{1}{18}\int_0^6 x^2dx=\frac{1}{18}\cdot\frac{1}{3}x^3\Big|_0^6$

$$=\frac{1}{18}\cdot\frac{1}{3}\cdot 6^3$$

2. $\because x_{0.5}=Min\{x|F_X(x)\ge 0.5\}$

$$F_X(x)=\int_0^x\frac{t}{18}dt=\frac{1}{18}\cdot\frac{1}{2}t^2\Big|_0^x\ge 0.5$$

$$\Rightarrow \frac{1}{36}x^2\ge 0.5 \Rightarrow x^2\ge 18 \Rightarrow x\ge 3\sqrt{2} \text{ or } x\le -3-\sqrt{2}(\text{不合})$$

$$\therefore x_{0.5}=Min\{x|x\ge 3\sqrt{2}\}\Rightarrow x_{0.5}=3\sqrt{2}$$

Ex 3(例 2.15) $X \sim pdf \quad f(x)=\sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2}} \quad , \ x>0$

1. $P(X>0)=1$ 2. $E(X)=?$ 3. $Var(X)=?$

$\Rightarrow$

1. $\int_0^\infty f(x)dx=\int_0^\infty \sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2}}dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{\frac{-x^2}{2}}dx$

令$u=\frac{x^2}{2} \ \Rightarrow \ du=\left(\frac{1}{2}\right)\cdot(2)xdx=xdx$

$\therefore \ \int_0^\infty e^{\frac{-x^2}{2}}dx=\int_0^\infty e^{-u}\cdot x^{-1}du=\int_0^\infty e^{-u}\cdot\frac{1}{\sqrt{2}}\cdot u^{-\frac{1}{2}}du$

$=\frac{1}{\sqrt{2}}\int_0^\infty u^{-\frac{1}{2}}e^{-u}du$

**Note：$P(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx \quad P(\alpha)=(\alpha-1)P(\alpha-1)$

$$\therefore \int_0^\infty u^{-\frac{1}{2}} e^{-u} du = P\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore \int_0^\infty f(x) dx = \sqrt{\tfrac{2}{\pi}} \cdot \tfrac{1}{\sqrt{2}} \cdot \sqrt{\pi} = 1$$

2. $E(X) = \int_{-\infty}^\infty xf(x) dx = \int_0^\infty x\sqrt{\tfrac{2}{\pi}} e^{-\frac{x^2}{2}} dx = \sqrt{\tfrac{2}{\pi}} \int_0^\infty xe^{-\frac{x^2}{2}} dx$

令 $u = \frac{x^2}{2} \;\Rightarrow\; du = \left(\tfrac{1}{2}\right) \cdot (2) xdx = xdx$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty e^{-u} du = \sqrt{\tfrac{2}{\pi}} \left(-e^{-u}\right)\Big|_0^\infty = \sqrt{\tfrac{2}{\pi}} \cdot (1) = \sqrt{\tfrac{2}{\pi}}$$

3. $E\left(X^2\right)=\int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2 \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^2 e^{-\frac{x^2}{2}} dx$

令 $u = \frac{x^2}{2} \;\Rightarrow\; du = \left(\frac{1}{2}\right)\cdot(2)xdx = xdx$

$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} 2ue^{-u} x^{-1} du = \sqrt{\frac{2}{\pi}} \int_0^{\infty} 2ue^{-u} \cdot \frac{1}{\sqrt{2}} u^{-\frac{1}{2}} du = 2\sqrt{\frac{1}{\pi}} \int_0^{\infty} u^{\frac{1}{2}} e^{-u} du$

$= 2\sqrt{\frac{1}{\pi}} P\left(\frac{3}{2}\right) = 2\sqrt{\frac{1}{\pi}} P\left(1+\frac{1}{2}\right) = 2\sqrt{\frac{1}{\pi}} \cdot \left(\frac{1}{2}\right) \cdot P\left(\frac{1}{2}\right)$

$= 2\sqrt{\frac{1}{\pi}} \cdot \left(\frac{1}{2}\right) \cdot \sqrt{\pi} = 1$

$\therefore\; Var(X) = E\left(X^2\right) - \left(E(X)\right)^2 = 1 - \left(\sqrt{\frac{2}{\pi}}\right)^2 == 1 - \frac{2}{\pi}$

Ex 4(例 2.16) A *r.v.* X with *pdf* $f(x)=e^{x-2}$, $x<2$.

Find the 75^{th} quantity of X.

$\Rightarrow$

$$x_q = Min\{x|F_X(x)\geq q\}$$

$$\because \ F_X(x)=P(X\leq x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{x} e^{t-2}dt=e^{t-2}\Big|_{-\infty}^{x}=e^{x-2}$$

$$\therefore \ x_{0.75}=Min\{x|e^{x-2}\geq 0.75\}$$

$$\because \ e^{x-2}\geq 0.75 \ \Rightarrow \ x-2\geq \ell n(0.75) \ \Rightarrow \ x\geq 2+\ell n(0.75)$$

$$\therefore \ x_{0.75}=Min\{x|x\geq 2+\ell n(0.75)\} \ \Rightarrow \ x_{0.75}=2+\ell n(0.75)$$

Ex 5(例 2.17) X is a continuous *r.v.* with *pdf* $f(x)$, $x > 0$

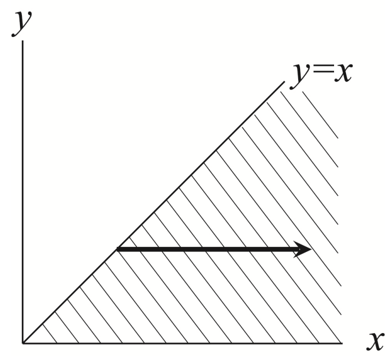
1. Show that $E(X) = \int_0^{\infty} [1 - F(x)] dx$
2. X with *pdf* $f(x) = xe^{-\lambda x}$, $x > 0$. Use (1) to find $E(X)$

$\Rightarrow$

1. $E(X) = \int_0^{\infty} x f(x) dx = \int_0^{\infty} \left[\int_0^x 1 dy \right] f(x) dx$

$$= \int_0^{\infty} \left[\int_0^x f(x) dy \right] dx = \int_0^{\infty} \left[\int_{x=y}^{\infty} f(x) dx \right] dy$$

$$= \int_0^{\infty} [1 - F(y)] dy$$



2. $\because \quad F(x)=\int_0^x f(t)dt=\int_0^x \lambda e^{-\lambda t}dt=\lambda\left(-\tfrac{1}{\lambda}e^{-\lambda t}\right)\Big|_{t=0}^{t=x}$

$$=1-e^{-\lambda x} \quad , \; x>0$$

$$\therefore \quad E(X)=\int_0^\infty\left[1-F(x)\right]dx=\int_0^x e^{-\lambda x}dx=-\frac{1}{\lambda}e^{-\lambda x}\Bigg|_0^\infty=\frac{1}{\lambda}$$

Ex 6(例 2.24)　X 的 *cdf* 為 $F_X(x)=\dfrac{x^2}{36}$ 求 $f_X(x)$ ，$E(X)$, $Var(X)$

$\Rightarrow$

$F_X(0)=0$ ，$F_X(6)=1$　　$\therefore$ x 的 range 為 $0\le x\le 6$

$$f_X(x)=\frac{d}{dx}F(x)=\frac{d}{dx}\cdot\frac{x^2}{36}=\frac{x}{18}$$

$$\therefore\quad f_X(x)=\begin{cases}\frac{x}{18} & ,\ 0\le x\le 6\\ 0 & ,\quad o.w.\end{cases}$$

$$E(X)=\int_0^6 xf(x)dx=\int_0^6 x\tfrac{x}{18}dx=\tfrac{1}{18}\int_0^6 x^2dx=\tfrac{1}{18}\cdot\tfrac{1}{3}x^3\Big|_0^6=4$$

$$E(X^2)=\int_0^6 x^2f(x)dx=\int_0^6 x^2\tfrac{x}{18}dx=\tfrac{1}{18}\int_0^6 x^3dx=\tfrac{1}{18}\cdot\tfrac{1}{4}x^4\Big|_0^6=18$$

$$Var(X)=E(X^2)-(E(X))^2=18-16=2$$

Ex 7(例 2.28) (連續與間斷混合型) 隨機變數 X , *cdf* 為

$$F_X(x)=\begin{cases} 0 & , \quad x<-1 \\ 0.3+0.2(x+1) & , \quad -1\le x<0 \\ 0.7+0.3x & , \quad 0\le x<1 \\ 1 & , \quad x\ge 1 \end{cases}$$

1. $P(X=-1)$ 　　2. $P(-0.6<X<0)$

3. $P(X=1)$ 　　4. $E(X)=?$ 　　$Var(X)=?$

$\Rightarrow$

1. $P(X=-1)=F_X(-1)-F_X(-1^-)=0.3+0.2(-1+1)-0=0.3$

2. $P\left(-0.6 < X < 0\right) = P\left(X < 0\right) - P\left(X \le -0.6\right) = F_X\left(0^-\right) - F_X\left(-0.6\right)$

$$= 0.3 + 0.2(0+1) - [0.3 + 0.2(-0.6+1)]$$

$$= 0.2 - 0.08 = 0.12$$

3. $P\left(X = 1\right) = F_X\left(1\right) - F_X\left(1^-\right) = 1 - \left(0.7 + 0.3\right) = 0$

4. $E\left(X\right) = \sum_x x\left[F\left(x\right) - F\left(x^-\right)\right] + \int_{-\infty}^{\infty} x dF_X\left(x\right)$

$$= \left(-1\right)\left[F\left(x\right) - F\left(x^-\right)\right] + 0[\quad] + 1\left[F\left(x\right) - F\left(x^-\right)\right] + \int_{-\infty}^{\infty} x dF_X\left(x\right)$$

$$= (-1)[0.3 - 0] + 1[1 - 1] + \int_{-\infty}^{\infty} x dF_X(x)$$

$$= -0.3 + \int_{-\infty}^{\infty} x dF_X(x)$$

$$= -0.3 + \int_{-\infty}^{\infty} x f_X(x) dx$$

$$\because \quad f_X(x) = \frac{d}{dx} F(x) = \begin{cases} 0.2 & , \quad -1 < x < 0 \\ 0.3 & , \quad 0 < x < 1 \\ 0 & , \quad o.\text{w.} \end{cases}$$

$$\therefore \quad \int_{-\infty}^{0} x f_X(x) dx = \int_{-1}^{0} x(0.2) dx + \int_{0}^{1} x(0.3) dx = 0.2 \cdot \tfrac{1}{2} x^2 \Big|_{0}^{1}$$

$$= 0.1(0 - 1) + 0.15 = 0.05$$

$$\therefore \quad E(X) = -0.3 + 0.05 = -0.25$$

$$E\left(X^2\right)=\sum_x x^2\left[F(x)-F\left(x^-\right)\right]+\int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$=(-1)^2\left[F(-1)-F\left(-1^-\right)\right]+1^2\left[F(1)-F\left(1^-\right)\right]+\int_{-1}^{0} x^2(0.2)\,dx+\int_{0}^{1} x^2(0.3)\,dx$$

$$=(0.3-0)+(1-1)+0.2\int_{-1}^{0} x^2\,dx+0.3\int_{0}^{1} x^2\,dx=\frac{7}{15}$$

$$Var(X)=E\left(X^2\right)-\left(E(X)\right)^2=\frac{7}{15}-(-0.25)^2$$

Ex 8(例 2.29) X 機率分配為 $f_X(x)=\begin{cases}0.2 & , \quad x=1\\ 0.5 & , \quad 1<x<2\\ 0.3 & , \quad x=2\end{cases}$

1. 求 cdf F_X 並畫圖　2. X 之中位數　3. 平均數

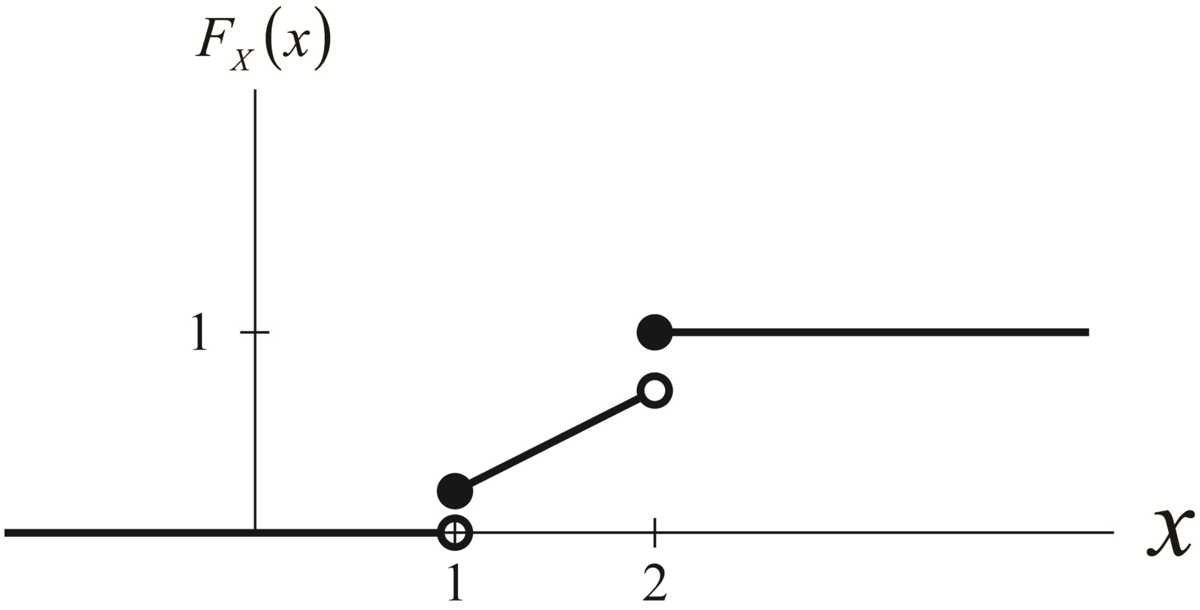
(連續與間斷混合型)

$\Rightarrow$

1. $F_X(x)=P(X\le x),\quad x\in\Re$

$$=\begin{cases}0 & , \quad x<1\\ 0.2+\int_1^x f_X(t)\,dt & , \quad 1\le x<2 \quad \Rightarrow \quad 0.2+\int_1^x 0.5dt=0.2+0.5(x-1)\\ 1 & , \quad x\ge 2\end{cases}$$

$$= \begin{cases} 0 & , \quad x < 1 \\ 0.5x - 0.3 & , \quad 1 \le x < 2 \\ 1 & , \quad x \ge 2 \end{cases}$$



2. $x_{0.5} = \min\{x \mid F_X(x) \geq 0.5\}$

$$0.5x - 0.3 \geq 0.5 \;\Rightarrow\; 0.5x \geq 0.8 \;\Rightarrow\; x \geq \frac{8}{5}$$

$$\therefore\; x_{0.5} = \min\left\{x \,\middle|\, F_X(x) \geq \frac{8}{5}\right\} \;\Rightarrow\; x_{0.5} = \frac{8}{5}$$

3. $E(X) = \sum_x x\left[F(x) - F(x^-)\right] + \int_1^2 xf(x)\,dx$

$$= 1\cdot(0.2) + 2\cdot(1-0.7) + \int_1^2 x(0.5)\,dx$$

$$= 0.2 + 0.6 + 0.5\cdot\frac{1}{2}x^2\Big|_1^2 = 1.55$$

Ex 9(例 2.32) A r.v. X with cdf $F_X(x)=\begin{cases} 0 & , \quad x<0 \\ 1-\left(\frac{2}{3}\right)e^{-x} & , \quad x\ge 0 \end{cases}$

Find $Var(x)$ (連續與間斷混合型)

$\Rightarrow$

$$\because P(X=0)=F(0)-F(0^-)=1-\left(\frac{2}{3}\right)-0=\frac{1}{3}$$

X 為混合型 *r.v.*

$$E(X)=\sum_x x\left[F(x)-F(x^-)\right]+\int_0^\infty xf(x)dx$$

$$=0\left[\frac{1}{3}-0\right]+\int_0^\infty x\left(\frac{2}{3}\right)e^{-x}dx$$

$$=\frac{2}{3}\Gamma(2)=\frac{2}{3}\big(1\cdot\Gamma(1)\big)=\frac{2}{3}$$

$$E\left(X^{2}\right)=\sum_{x}x^{2}\left[F(x)-F\left(x^{-}\right)\right]+\int_{-\infty}^{\infty}x^{2}f_{X}(x)\,dx$$

$$=\int_{0}^{\infty}x\left(\frac{2}{3}\right)e^{-x}dx=\frac{2}{3}\int_{0}^{\infty}xe^{-x}dx=\frac{2}{3}\Gamma(3)=\frac{2}{3}\cdot 2\cdot\Gamma(2)$$

$$=\frac{2}{3}\cdot 2\cdot 1\cdot\Gamma(1)=\frac{4}{3}$$

$$\therefore\ Var(X)=E\left(X^{2}\right)-\big(E(X)\big)^{2}$$

$$=\frac{4}{3}-\left(\frac{2}{3}\right)^{2}=\frac{8}{9}$$

Ex 10(例 2.33) A continuous *r.v.* X with *pdf*

$$f_X(x)=\begin{cases} k(1-x)x^2 & , \quad 0<x<1 \\ 0 & , \quad o.w. \end{cases}$$

1. Find k. 2. Find $f\left(x \middle| x>0.5\right)$ 3. Find $E\left(X \middle| X>0.5\right)$

$\Rightarrow$

1. $\int_{-\infty}^{\infty} f(x)dx=1 \Rightarrow k\int_0^1 (1-x)x^2 dx=1 \Rightarrow k\int_0^1 \left(x^2-x^3\right)dx=1$

$$\Rightarrow k\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=1 \Rightarrow k\left(\frac{1}{3}-\frac{1}{4}\right)=1$$

$$\Rightarrow \frac{k}{12}=1 \Rightarrow k=12$$

$$\therefore \quad f_X(x) = \begin{cases} 12(1-x)x^2 & , \quad 0 < x < 1 \\ 0 & , \quad o.w. \end{cases}$$

$$2. \quad f_X(x \mid x > 0.5) = \frac{f(x)}{P(x > 0.5)} = \frac{12(1-x)x^2}{\frac{11}{16}} \qquad , \quad 0.5 < x < 1$$

$$P(x > 0.5) = \int_{0.5}^{1} 12(1-x)x^2 dx = 12\int_{0.5}^{1} \left(x^2 - x^3\right) dx = \frac{11}{16}$$

$$\therefore \quad f_X(x \mid x > 0.5) = \begin{cases} \frac{192}{11}(1-x)x^2 & , 0.5 < x < 1 \\ 0 & , o.w. \end{cases}$$

3. $E\left(X \mid X > 0.5\right) = \int_{0.5}^{1} x f\left(x \mid x > 0.5\right) dx = \int_{0.5}^{1} x \cdot \frac{192}{11}\left(1 - x\right) x^2 dx$

$$= \frac{192}{11}\int_{0.5}^{1} x^3\left(1 - x\right) dx = \frac{192}{11}\int_{0.5}^{1}\left(x^3 - x^4\right) dx$$

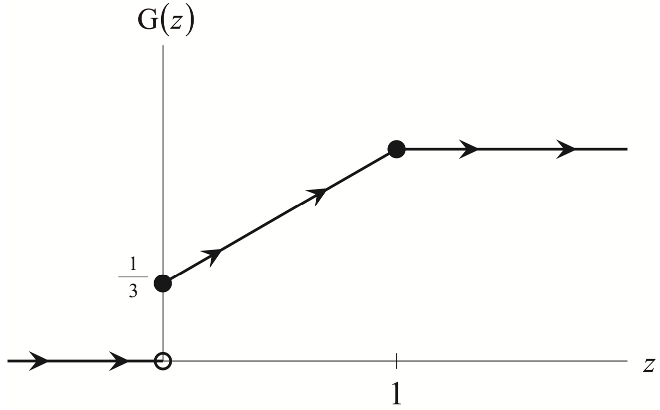
$$= \frac{39}{55}$$

Ex 11(例 3.7) A *r.v.* Z with *cdf* $G(z)=\begin{cases} 0 & , \quad z<0 \\ \left(\dfrac{2z+1}{3}\right) & , \quad 0\le z<1 \\ 1 & , \quad o.w. \end{cases}$

Get the *mgf* of Z be $m(t)=E\left(e^{tz}\right)$. Find $m(1)$

$\Rightarrow$

Z 為混合型



$$m(t) = E\left(e^{tz}\right) = \sum_{z \geq 0} e^{tz}\left[F(z) - F\left(z^{-}\right)\right] + \int_{-\infty}^{\infty} e^{tz} f(z)\, dz$$

$$= 1 \cdot \left[\frac{1}{3} - 0\right] + \int_0^1 e^{tz} \cdot \frac{2}{3} dz = \frac{1}{3} + \frac{2}{3}\int_0^1 e^{tz} dz = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{t} e^{tz}\Big|_{z=0}^{z=1}$$

$$= \frac{1}{3} + \frac{2}{3t}\left(e^{t} - 1\right)$$

$$m(1) = \frac{1}{3} + \frac{2}{3}\left(e^{1} - 1\right)$$

$$= \frac{2}{3} e - \frac{1}{3}$$

Ex 12(例 4.6) A *r.v.* X with $E(X)=33,\ Var(X)=16$

1. What is the lower bound for $P(23 < X < 43)$

2. What is the upper bound for $P(|X-33| \geq 14)$

$\Rightarrow$

1. $\because\ P(|X-\mu| > k\sigma) \leq \frac{1}{k}$ or $P(|X-\mu| \leq k\sigma) > 1-\frac{1}{k^2}$

$$P(23 < X < 43) = P(23-33 < X-33 < 43-33) = P(-10 < X-\mu < 10)$$

$$= P(|X-\mu| < 10) = P(|X-\mu| < k(4)) > 1-\frac{1}{k^2}$$

$$= 1-\frac{1}{\left(\frac{5}{2}\right)^2} = 1-\frac{1}{\frac{25}{4}} = 1-\frac{4}{25} = \frac{21}{25} = 0.84$$

2. $P\left(|X-33|\geq 14\right)=P\left(|X-\mu|\geq k(4)\right)\leq \frac{1}{k^2}$

$$=\frac{1}{\left(\frac{7}{2}\right)^2}=\frac{4}{49}=0.082$$