第四章 機率基礎概念

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第四章　單變量隨機變數

隨機變數(random variable） $X:\Omega\rightarrow\Re$　　*i.e.* X是由樣本空間Ω映射到實數的函數。

$$\forall \underbrace{\omega}_{outcome} \in \Omega, \quad \underbrace{X(\omega)}_{\text{隨機變數X的實現值}(realsation)} \quad \text{為實數。}$$

ω由隨機試驗所決定，ω未確定前$X(\omega)$無法確定。一旦ω確定後, $X(\omega)$才確定。

$\therefore$ X是依據隨機機率而產生不同的實現值的函數。稱為 random variable　or　單變量 random variable (r.v.)。

Ex 4.1 擲一個骰子，定義 $r.v.\ X_1, X_2, X_3$如下：

$$X_1(\omega)=\begin{cases}1 & if\ \omega=1\\ 2 & if\ \omega=2\\ \vdots & \quad\vdots\\ 6 & if\ \omega=6\end{cases}$$

$$X_2(\omega)=\begin{cases}6 & if\ \omega=1\\ 7 & if\ \omega=2\\ \vdots & \quad\vdots\\ 11 & if\ \omega=6\end{cases}\qquad i.e.\ X_2=X_1+5.$$

$$X_2(\omega)=\begin{cases}5 & if\ \omega=1\\ 10 & if\ \omega=2\\ \vdots & \quad\vdots\end{cases}\qquad i.e.\ X_3=5X_1$$

第4.1節　離散的隨機變數

discrete *r.v.* X ：一個 *r.v.* X，實現值為有限多個(finite)或為可數無限多個(countable infinite)稱之。

Ex.

在 **Ex 4.1** 中 X_1, X_2, X_3 為 discrete *r.v.*。

某一個時段通過收費站車子的數量。

利用 *r.v.* 定義事件(event)

Ex. 在 **Ex 4.1** 中

$\{\omega : X_1(\omega) \le 2\} = \{1, 2\}$ *i.e.*骰子出現 1 點與 2 點。

$\{\omega : 7 < X_2(\omega) \le 9\} = \{3, 4\}$

$\{\omega : X_3(\omega) = 25\} = \{5\}$

X： discrete $r.v.$ 其值為$b_1, b_2, \cdots$ 定義X的機率函數(probability function) f_X如下：

$$f_X(b_i) = P(\{\omega \in \Omega \mid X(\omega) = b_i\}) \quad \forall i = 1, 2, \cdots$$

而在其他值時 $f_X = 0$

將$\{\omega \in \Omega \mid X(\omega) = b_i\}$ 簡寫成 $\{X = b_i\}$

而$\{\omega \in \Omega \mid X(\omega) \le a\}$ 簡寫成 $\{X \le a\}$

假設 discrete $r.v.$ X，則$\{c < X \le a\}$為包含各種 event such that $\{X = b_i, c < b_i \le a\}$

$$i.e.\ \{c < X \le a\} = \bigcup_{c<b_i\le a}\{X = b_i\}$$

$\because \{X = b_i\}$ $\forall i$ 為互斥事件

$$\therefore P(\{c < X \le a\}) = P\left(\bigcup_{c<b_i\le a}\{X = b_i\}\right) = \sum_{c<b_i\le a} P(X = b_i)$$

$$= \sum_{c<b_i\le a} f_X(b_i)$$

另外，$\Omega = \bigcup_i \{X = b_i\}$

$$\Rightarrow \quad P(\Omega) = P\left(\bigcup_i \{X = b_i\}\right) = \sum_i P(\{X = b_i\}) = \sum_i f_X(b_i)$$

$\therefore$ 若 f_X 為機率函數，必滿足以下性質：

(1) $f_X(b_i) \geq 0 \quad \forall i = 1,2,\cdots$

(2) $\sum_i f_X(b_i) = 1$

另外一種表示法：

$$\forall x \in \Re,\ \ f_X(x) = P(\{\omega \in \Omega | X(\omega) = x\})$$
$$= P(X = x)$$

而 $P(\{c < X \le a\}) = P(\{\omega \in \Omega | c < X(\omega) \le a\})$

$$= P(\{\omega \in \Omega | \mathrm{X}(\omega) = x,\ c < x \le a\})$$
$$= P\left(\bigcup_{c<x\le a} \{\omega \in \Omega | \mathrm{X}(\omega) = x\}\right)$$
$$= \sum_{c<x\le a} P(\{\omega \in \Omega | \mathrm{X}(\omega) = x\})$$
$$= \sum_{c<x\le a} P(X = x) = \sum_{c<x\le a} f_X(x)$$

Ex 4.2 考慮 Ex 4.1 中 X_1

$$X_1(\omega)=\begin{cases}1 & if\ \omega=1\\ 2 & if\ \omega=2\\ \vdots & \vdots\\ 6 & if\ \omega=6\end{cases}$$

$$f_X(1)=P(\{\omega\in\Omega|X_1(\omega)=1\})=P(\{\omega=1\})=\frac{1}{6}$$

$$\vdots$$

$$f_X(6)=P(\{\omega\in\Omega|X_1(\omega)=6\})=P(\{\omega=6\})=\frac{1}{6}$$

$$P(2 < X_1 \leq 4) = P(\{\omega \in \Omega | 2 < X_1(\omega) \leq 4\})$$
$$= P(\{3,4\}) = \frac{2}{6} = \frac{1}{3}$$
$$= P(\{3\} \cup \{4\}) = P(\{3\}) + P(\{4\})$$
$$= f_X(3) + f_X(4) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$P(\{X_1 \leq 2\}) = f_X(1) + f_X(2) = \frac{1}{3}$$
$$P(\{X_1 > 3\}) = f_X(4) + f_X(5) + f_X(6) = \frac{1}{2}$$

X： discrete *r.v.*, 定義其**累積分配函數(Cumulative Distribution function , cdf)**, 以 F_X 表示如下：

$$\forall\ x \in \Re,\ F_X(x) = P(X \le x) = P\left(\left\{\omega \in \Omega \middle| X(\omega) \le x\right\}\right)$$

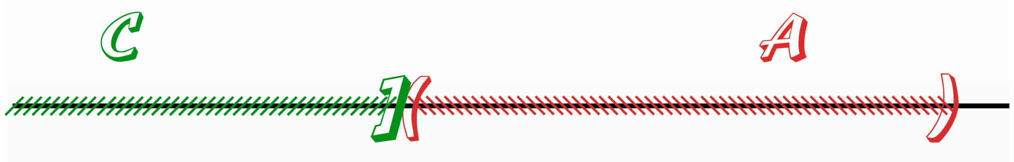
$$= P\left(\left\{\omega \in \Omega \middle| X(\omega) = y,\ \ y \le x\right\}\right)$$

$$= P\left(\bigcup_{y \le x} \left\{X(\omega) = y\right\}\right)$$

$$= \sum_{y \le x} P\left(X(\omega) = y\right)$$

$$= \sum_{y \le x} f_X(y)$$

i.e. 給定一個實數 x, $F_X(x)$: 將 X 實現值小於等於 x 的機率相加



假設$c < a$

則 $\{X \le a\} = \{X \le c\} \cup \{c < X \le a\}$

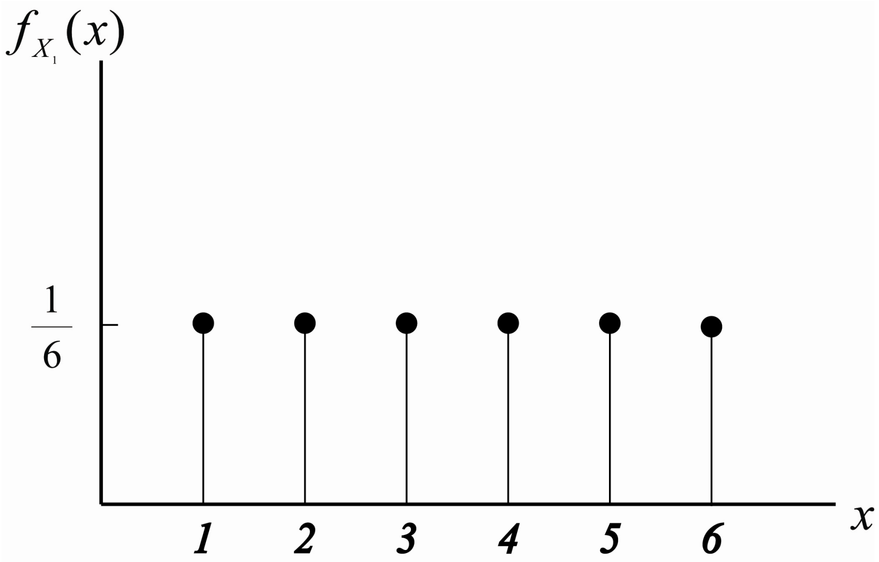
$$\Rightarrow P(\{X \le a\}) = P(\{X \le c\}) + P(\{c < X \le a\})$$

$$\Rightarrow P(c < X \le a) = P(X \le a) - P(X \le c)$$

$$= F_X(a) - F_X(c)$$

Ex 4.3 考慮 Ex 4.1 中 X_1

$\because$ X_1 的機率函數圖形如下：



$$F_{X_1}(-2) = P(X_1 \le -2) = P(\phi) = 0$$

$$F_{X_1}(0.87) = P(X_1 \le 0.87) = P(\phi) = 0$$

X_1的 cdf 為:

$$if \quad x<1 \quad \Rightarrow \quad F_{X1}(x)=P(X_1 \le x)=0$$

$$if \quad 1 \le x < 2 \quad \Rightarrow \quad F_{X1}(x)=P(X_1 \le x)=\frac{1}{6}$$

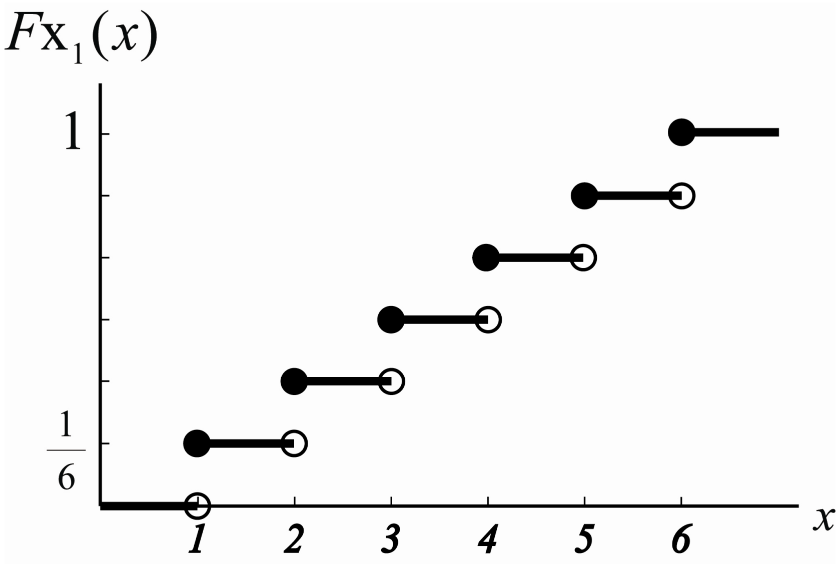
$$if \quad 2 \le x < 3 \quad \Rightarrow \quad F_{X1}(x)=P(X_1 \le x)=\frac{2}{6}$$

$$\vdots$$

$$if \quad 5 \le x < 6 \quad \Rightarrow \quad F_{X1}(x)=P(X_1 \le x)=\frac{5}{6}$$

$$if \quad x \ge 6 \quad \Rightarrow \quad F_{X1}(x)=P(X_1 \le x)=1$$

$$\therefore F_{X1}(x) = \begin{Bmatrix} 0 & , if & x < 1 \\ \frac{1}{6} & , if & 1 \le x < 2 \\ \frac{2}{6} & , if & 2 \le x < 3 \\ \vdots & & \vdots \\ 1 & , if & x \ge 6 \end{Bmatrix}$$



$\therefore\ F_{X1}$ **為一階梯函數 (Step Function)**

$$P(X_1 = 2) = \frac{1}{6} = F_{X1}(2) - F_{X1}(2^-)$$

In general, $P(X_1 = a) = F_{X1}(a) - F_{X1}(a^-)$

Ex. $P(X_1 = 2.8) = F_{X1}(2.8) - F_{X1}(2.8^-) = \frac{2}{6} - \frac{2}{6} = 0$

Notes： If $F(x)$ is a cdf , then

1. $F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0$

2. $F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1 \, F_X(\infty) = \lim_{x\to\infty} \ell n \, F_X(x) = 1$

3. $F_X(x)$ is a nondecreasing function.

   i.e. $if \quad x_1 < x_2 \quad \Rightarrow \quad F_X(x_1) \le F_X(x_2)$

4. F_X must be right continuous.

   i.e. $\lim_{x\to a^+} F_X(x) = F_X(a) \quad \forall \quad a \in \Re$